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On the geometry of super Yang-Mills theories: phases and irreducible polynomials

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Abstract: We study the algebraic and geometric structures that underly the space of vacua of $\mathcal{N} = 1$ super Yang-Mills theories at the non-perturbative level. Chiral operators are shown to satisfy polynomial equations over appropriate rings, and the phase structure of the theory can be elegantly described by the factorization of these polynomials into irreducible pieces. In particular, this idea yields a powerful method to analyse the possible smooth interpolations between different classical limits in the gauge theory. As an application in $\mathrm{U}(N)$ theories, we provide a simple and completely general proof of the fact that confining and Higgs vacua are in the same phase when fundamental flavors are present, by finding an irreducible polynomial equation satisfied by the glueball operator. We also derive the full phase diagram for the theory with one adjoint when $N \leq 7$ using computational algebraic geometry programs.

Keywords: Supersymmetric gauge theory, Nonperturbative Effects, Differential and Algebraic Geometry.

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Contents

1. Introduction

1.1 General presentation

The study of the non-perturbative aspects of $\mathcal{N} = 1$ supersymmetric gauge theories has revealed over the years many remarkable physical phenomena that can be described in a rich mathematical framework. The fundamental tool is the existence of special *chiral* operators that preserve half of the supercharges. The expectation values of these operators are space-time independent and depend *holomorphically* on the various parameters of the theory. This allows to use the many tools of complex analysis, in particular it is possible to make analytic continuations to derive results at strong coupling from semi-classical instanton calculations. Recently, a completely general and first-principle approach has been developed to compute any chiral operator expectation values along these lines [1–4]. The results have an interpretation in the context of the open/closed string duality (geometric transitions, matrix models), brane engineering, mirror symmetry, integrable systems etc...They lie at the heart of many developments in Quantum Field Theory and String Theory over the last 15 years.

In the present work we are going to formulate the results in an algebraic and geometric language that turns out to be extremely natural and efficient to understand the general structure of the theory and to derive the physical consequences of the solutions. In particular, we revisit some fundamental notions like the chiral ring, whose full significance has not been fully understood and exploited in previous works. We also correct some confusions that have appeared in the literature.

This research is motivated by the fact that a satisfactory understanding of the global properties of the space of vacua of supersymmetric theories, including the phase structure and the possible interpolations between different classical vacua, requires new powerful computational tools. The framework that we are going to develop allows to reduce many interesting physical questions to simple arithmetic properties of polynomials. Moreover, when necessary, our approach lends itself very well to calculations on the computer.

An important conceptual issue is to understand the nature of the various phases in which the gauge theories can be realized. For example, is it possible to distinguish the phases using some symmetry principle? This is an outstanding open problem. The standard 't Hooft's and Wilson's order parameters provide a partial answer, but it is known that they fail to provide a complete classification [5]. The results of our work can be used to shed a new interesting light on these questions, as will be explained in a separate publication [6].

1.2 Vacua versus phases

One of the most interesting aspect of supersymmetric gauge theories is to have a very rich and complex landscape of vacua. The number of vacua can be very large, growing exponentially with the number of colours. We shall be able to study examples with several thousands of vacua in the following. The vacua realized in a given theory can have very different physics, with various particle spectra and gauge groups. The structure has actually many similarities with the M/string theory landscape.

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The notion of vacuum is very central in the usual approaches to quantum field theory (and actually to any quantum theory). One of the basic reason is that quantum mechanics is usually formulated by starting from a classical solution (a classical vacuum) and then quantizing around this solution. Typically, one expands the observables in powers of a parameter measuring the strength of the quantum corrections around the classical solution under consideration. In essence, this is an *analytic* approach. In the favorable cases where the expansion converges (this is what happens in the chiral sector of the theory), one can then have access to the genuine quantum regime. The resulting analytic formulas can be very cumbersome and the underlying strongly quantum physics can be hard, if not impossible, to describe.

On the other hand, from a purely quantum point of view, independently of any semi-classical approximation, the notion of a classical (or quantum) vacuum is peripheral. This fundamental fact will become clearer and clearer the further we advance in the paper. The central invariant concept is the one of *phase*. A precise definition will be given later, but we can already describe the most relevant features. A given gauge theory may be realized in various phases, but the main property of individual phases is that by varying the parameters in arbitrary ways the theory always remains in the same phase. In this sense, a phase can be considered to be by itself a consistent quantum theory. Many vacua can belong to the same phase, which means equivalently that a given phase can have many different classical limits. Any classical limit in a given phase can be obtained from any other classical limit in the same phase by a suitable analytic continuation. These analytic continuations can be strongly quantum mechanical, involving highly non-trivial effects like the exchange between D-brane like objects and solitonic branes and the changing of the unbroken gauge groups [5, 7−10].

It is when one wishes to study the phases in a fully quantum way, in particular taking into account all the possible classical limits at the same time, that the algebro-geometric approach that we shall use is very powerful.

1.3 Algebraic geometry

The geometric picture is actually very simple. It is known that supersymmetry implies that the space of vacua $\mathscr{M}$ must be a complex manifold. This is particularly clear at the classical level, where the classical space of vacua $\mathscr{M}_{\mathrm{cl}}$ is described by the F-term constraints on the set of gauge invariant chiral operators. The variety $\mathscr{M}_{\mathrm{cl}}$, even though it doesn't know about the strongly coupled gauge dynamics, can be quite non-trivial and interesting as recent works have shown [11]. In section 2, we are going to explain in details how to define the quantum algebraic variety $\mathscr{M}$, describing explicitly its defining equations. The ring of chiral observables of the theory coincides with the ring of functions defined on the variety. A crucial aspect is that the variety $\mathscr{M}$ is not in general irreducible. The existence of distinct phases $|\varphi)$ in the gauge theory precisely corresponds to the decomposition of $\mathscr{M}$ into irreducible components,

$$\mathscr{M} = \bigcup_{|\varphi)} \mathscr{M}_{|\varphi)} \,. \tag{1.1}$$

Algebraically, a given irreducible factor $\mathscr{M}_{|\varphi)}$ is characterized by a set of special relations satisfied by the chiral operators in the phase $|\varphi)$, making the ideal of operator relations prime. In practice, this can be described by the factorization of certain polynomials defined over appropriate rings into irreducible pieces. An extremely simple description of the operator algebra in a given phase in terms of "primitive operators" can then be given. All these aspects are explained in section 3.

In the above picture, the vacua simply correspond to the intersection points between $\mathscr{M}$ and a set of hyperplanes that corresponds to fixing the parameters of the gauge theory to some special values. If v is the total number of vacua and p the total number of parameters, $\mathscr{M}$ can then be seen as a v-fold cover of $\mathbb{C}^p$. However this description is quite arbitrary. For example, one could slice $\mathscr{M}$ with generic hyperplanes. The number of intersection points, which is the degree of the variety, is then in general larger than v. On the other hand, the decomposition (1.1) expresses an intrinsic property of the space $\mathscr{M}$ and of the quantum gauge theory.

One advantage of the algebraic description of the space of vacua that we shall set up is that methods from computational algebraic geometry become available. This field has been developing rapidly over the last few years, with a profound impact on research in algebraic geometry and commutative algebra. A list of available softwares can be found in [12]. We have used both Singular (for symbolic computations) and PHC (for numerical computations) [13, 14]. These programs implement powerful algorithms that are able to compute the decomposition (1.1) into irreducible components, see section 5.

1.4 Applications

One outstanding application that we are going to study is the following. Consider a $\mathrm{U}(N)$ gauge theory with fields in the fundamental representation. In this case, a test charge in any representation of the gauge group can be screened by the dynamical fundamental fields, and the usual criteria used to distinguish the confining and the Higgs regimes do not work. In fact, it has been known for almost 30 years that the confining and Higgs regimes can be smoothly connected and are thus in the same phase when the theory is formulated on the lattice [15].

In the continuum, the problem is much more difficult to study because the interpolation cannot be described perturbatively or semi-classically. In [10], it was convincingly argued that the solutions to $\mathcal{N}=1$ supersymmetric gauge theories with fundamentals seemed to have the required features for describing a single Higgs/confining phase. A proof could not be given, however, because of the apparent complexity of the explicit solution of the model. One uses auxiliary algebraic curves and meromorphic functions with a complicated pole structure defined on these curves. To understand the phase structure one then has to study in great details how the algebraic curves and the poles are deformed when the parameters are varied. This is made extremely difficult and cumbersome by the fact that the curves and the positions of the poles must obey complicated non-linear constraints. This problem was further studied in [16] using very detailed calculations and numerical analysis in special cases.

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In our framework, the equivalence between the "Higgs" and "confining" phases follow from the fact that the corresponding vacua belong to the same irreducible component of the space of vacua. We shall be able to provide a completely general and simple proof of this fact in section 4, by finding an irreducible polynomial equation satisfied by the gluino condensate.

Another interesting model, that has been much studied in the literature, is the $\mathrm{U}(N)$ theory with only one adjoint matter chiral superfield. The landscape of vacua for this model is very interesting, with a highly non-trivial phase structure. We shall give a complete description of the space of vacua for all $N \leq 7$ in section 5, providing in particular many explicit and non-trivial examples of irreducible polynomial equations. For example, the $\mathrm{U}(7)$ theory can be realized in 10 distinct phases and a model that realizes all these phases must have at least 11075 vacua. The decomposition into phases is worked out by proving the irreducibility of several complicated polynomials of degrees up to 126.

1.5 Remarks and terminology

The aim of this paper is to develop a general framework in which the solutions of the theories can be naturally expressed and exploited. However, we do not explain how the explicit solutions are obtained. Let us simply stress that direct derivations from first principles are now available [1−3].

All the necessary algebraic notions are introduced in a pedagogical way and are motivated by physical questions. The tools we need are fairly elementary and do not go beyond the beginning graduate level. Excellent references that we have used are listed in [17].

A *field* in the following always refers to the notion of an algebraic field. A field is thus a commutative ring in which every non-zero element has an inverse. A basic result explained in section 2 is that in a given phase the ring of chiral operators of the theory is actually a field, i.e. every non-zero operator has an inverse.

If k is a field, we denote by $k[X_1,\ldots,X_n]$ the ring of polynomials with n indeterminates $X_1,\ldots,X_n$ and coefficients in k. Thus the $X_1,\ldots,X_n$ are always unconstrained variables. On the other hand, we denote by $k[\mathcal{O}_1,\ldots,\mathcal{O}_n]$ the ring generated by arbitrary variables $\mathcal{O}_1,\ldots,\mathcal{O}_n$ over k. These variables may satisfy polynomial relations over k. If I is the ideal generated by these relations, then

$$k[\mathcal{O}_1,\ldots,\mathcal{O}_n] = k[X_1,\ldots,X_n]/I\,. \tag{1.2}$$

An ideal I is said to be *prime* if $ab \in I$ implies that either a or b is in I. The quotient ring (1.2) is then an integral domain and one can build a field of fractions from it in the same way as one builds the field of rational numbers $\mathbb{Q}$ from the ring of integers $\mathbb{Z}$.

2. Foundations

2.1 Generalities

We consider a general $\mathcal{N}=1$ supersymmetric gauge theory in four dimensions. The lowest components of gauge invariant chiral superfields are called *chiral operators*. Equivalently, chiral operators are local gauge invariant operators that commute (in the case of bosonic operators) or anticommute (in the case of fermionic operators) with the left-handed supersymmetry charges.

The Lie algebra $\mathfrak{g}$ of the gauge group decomposes into a direct sum of $\mathfrak{u}(1)$ factors and simple non-abelian factors,

$$\mathfrak{g} = \mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1) \oplus_\alpha \mathfrak{g}_\alpha \,. \tag{2.1}$$

To each non-abelian factor $\mathfrak{g}_\alpha$ is associated a complex gauge coupling constant

$$\tau_\alpha = \frac{\theta_\alpha}{2\pi} + i\frac{4\pi}{g_\alpha^2} \cdot \tag{2.2}$$

In the quantum theory, the gauge couplings run,

$$\tau_\alpha(\mu) = \frac{i\beta_\alpha}{2\pi} \ln \frac{\mu}{\Lambda_\alpha} \cdot \tag{2.3}$$

The coefficients β_α can be computed at one loop and the higher loop effects are included in the complex scales Λ_α. These scales, or more conveniently the instanton factors

$$q_\alpha = \Lambda_\alpha^{\beta_\alpha} = \mu^{\beta_\alpha} e^{2i\pi\tau_\alpha} \,, \tag{2.4}$$

can be interpreted as being the lowest components of background chiral superfields [18, 19]. The q_α will be denoted collectively by $\boldsymbol{q}$.

On top of the $\boldsymbol{q}$, the theory has parameters $\boldsymbol{g} = (g_k)$ that couple to chiral operators $\mathcal{O}_k$ in the tree-level superpotential,

$$W_{\text{tree}} = \sum_k g_k \mathcal{O}_k \,. \tag{2.5}$$

As for the $\boldsymbol{q}$, the parameters $\boldsymbol{g}$ are best viewed as background chiral operators.

A fundamental property of the expectation values of chiral operators is that they depend *holomorphically* on $\boldsymbol{g}$ and $\boldsymbol{q}$. *Solving* the theory means computing the analytic functions $\langle \mathcal{O} \rangle (\boldsymbol{g}, \boldsymbol{q})$ for all the chiral operators $\mathcal{O}$. We are going to describe some general properties of these analytic functions below.

2.2 On the number of vacua

2.2.1 With or without a moduli space

We are interested in models that do not break supersymmetry. For a generic superpotential (2.5), one typically finds a finite number of supersymmetric vacua. In some special cases, when (2.5) has flat directions that are not lifted in the quantum theory, there is a moduli space of vacua.

A theory with a moduli space can often we obtained from the more generic case without a moduli space by turning off certain parameters in (2.5). In this situation, the solution with a moduli space is a special case of the solution with a finite number of vacua. Independently of this observation, it turns out that the cases with and without a moduli space can

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be formally studied along the same lines. This can be easily understood as follows. A moduli space of dimension d can be parametrized by d coordinates that correspond to the expectation values of d massless chiral operators $\mathcal{O}_1, \ldots, \mathcal{O}_d$.[1] Once the parameters of the theory and the $\langle \mathcal{O}_i \rangle$ are fixed, all the other expectation values are unambiguously determined, up to a possible finite degeneracy. If we treat the $\langle \mathcal{O}_i \rangle$ for $1 \leq i \leq d$ as the other parameters $\boldsymbol{g}$ and $\boldsymbol{q}$, the solution can then be described as in the case of the theories with a finite number of vacua.

For the above reasons and if not explicitly stated otherwise, we shall focus in the following on theories that have a finite number v of vacua.

2.2.2 Counting the vacua

Let $|i\rangle$, $1 \leq i \leq v$, be the supersymmetric vacua of the theory. Mathematically, the existence of multiple vacua is equivalent to the multi-valuedness of the analytic functions $\langle \mathcal{O} \rangle(\boldsymbol{g}, \boldsymbol{q})$. Each possible value $\mathcal{O}_i(\boldsymbol{g}, \boldsymbol{q})$ corresponds to the expectation in a vacuum $|i\rangle$,

$$\langle i|\mathcal{O}|i\rangle = \mathcal{O}_i(\boldsymbol{g}, \boldsymbol{q})\,. \tag{2.6}$$

The number of vacua is thus equal to the degree of the analytic functions $\langle \mathcal{O} \rangle(\boldsymbol{g}, \boldsymbol{q})$. This number cannot change when the parameters are varied, except at special points where the expectation values may go to infinity and the associated vacuum disappears from the spectrum.

From the above remarks it is easy to compute v explicitly in any given model by looking at the small q_α expansion of the expectation values, which can be straightforwardly obtained from the explicit solutions. At the classical level, $q_\alpha = 0$, the vacua are found by extremizing the tree-level superpotential (2.5). To each classical solution $|a\rangle_{\rm cl}$ is associated a certain pattern of gauge symmetry breaking. The Lie algebra $\mathfrak{h}^{|a\rangle_{\rm cl}}$ of the unbroken gauge group in $|a\rangle_{\rm cl}$ decomposes as

$$\mathfrak{h}^{|a\rangle_{\rm cl}} = \mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1) \oplus_\beta \mathfrak{h}_\beta^{|a\rangle_{\rm cl}}\,. \tag{2.7}$$

In each simple non-abelian factor $\mathfrak{h}_\beta^{|a\rangle_{\rm cl}}$, with associated dual Coxeter number $h^\vee(\mathfrak{h}_\beta^{|a\rangle_{\rm cl}})$, chiral symmetry breaking implies a $h^\vee(\mathfrak{h}_\beta^{|a\rangle_{\rm cl}})$-fold degeneracy. The number of quantum vacua associated to the classical solution (2.7) is thus given by $\prod_\beta h^\vee(\mathfrak{h}_\beta^{|a\rangle_{\rm cl}})$. The total number of vacua is then obtained by summing over all the classical solutions,

$$v = \sum_{|a\rangle_{\rm cl}} \prod_\beta h^\vee(\mathfrak{h}_\beta^{|a\rangle_{\rm cl}})\,. \tag{2.8}$$

We see in particular that v changes precisely when the number of classical solutions changes. This happens when some of the g_k in (2.5) vanish and the asymptotic behaviour of the tree-level superpotential is changed.

[1]As will become clear in the following, the moduli space may have various irreducible components corresponding to different phases of the theory. The dimension can vary from one component to the other and thus, strictly speaking, the discussion in this paragraph applies for each irreducible component independently.

Example 1. In the case of the pure gauge theory based on a simple gauge group G, $v = h^\vee(\mathfrak{g})$. For example, for $G = \mathrm{SU}(N)$, $v = N$. If $G = \mathrm{U}(N)$, one also has $v = N$, because the $\mathfrak{u}(1)$ factor in (2.7) does not change v.

Example 2. Let us consider the $\mathrm{U}(N)$ gauge theory, with N_f flavours of quarks corresponding to chiral superfields Q^{a}_f and $\tilde{Q}^f_{\mathsf{a}}$ in the fundamental and anti-fundamental representations respectively (a and a' are gauge indices and f and f' are flavour indices). Let us choose the tree-level superpotential to be

$$W_\mathrm{tree} = \tilde{Q} m Q = \sum_{1\leq f,f'\leq N_\mathrm{f}} \sum_{1\leq \mathsf{a}\leq N} \tilde{Q}^f_{\mathsf{a}} m_f^{\ f'} Q^{\mathsf{a}}_{f'}\,, \tag{2.9}$$

where $m = (m_f^{\ f'})$ is an invertible mass matrix. The classical solutions correspond to $Q = \tilde{Q} = 0$ and thus to an unbroken gauge group. The number of vacua is thus $v = N$. Physically, one can integrate out the quarks and find at low energy a pure $\mathrm{U}(N)$ gauge theory.

Example 3. Let us now consider the paradigmatic example of the $\mathrm{U}(N)$ gauge theory with one adjoint chiral superfield ϕ and tree-level superpotential

$$W_\mathrm{tree} = \mathrm{Tr}\, W(\phi)\,, \tag{2.10}$$

where W is a polynomial such that

$$W'(z) = \sum_{k=0}^{d} g_k z^k = g_d \prod_{i=1}^{d} (z - w_i)\,. \tag{2.11}$$

The classical solutions $|N_1,\ldots,N_d\rangle_\mathrm{cl}$ are labeled by non-negative integers $(N_1,\ldots,N_d)$ satisfying $\sum_{i=1}^d N_i = N$. The integer N_i corresponds to the number of eigenvalues of the matrix ϕ that are equal to w_i. The number v_cl of classical vacua is thus equal to the number of partitions of N by d non-negative integers,

$$v_\mathrm{cl} = \binom{N+d-1}{d-1} = \frac{(N+d-1)!}{(d-1)!N!}\,\cdot \tag{2.12}$$

To a given classical solution $|N_1,\ldots,N_d\rangle_\mathrm{cl}$, we associate an integer r that counts the number of non-zero N_i. We call r the *rank* of the solution (this terminology comes from the fact that the low energy gauge group in the quantum theory is $\mathrm{U}(1)^r$ in this case). Taking into account a trivial combinatorial factor $\binom{d}{r}$ corresponding to the choice of the non-zero positive integers N_i, there are

$$v_{\mathrm{cl},\,r} = \binom{d}{r}\binom{N-1}{r-1} \tag{2.13}$$

classical solutions of rank r, and obviously $v_\mathrm{cl} = \sum_{r=1}^{\min(d,N)} v_{\mathrm{cl},\,r}$.

The Lie algebra of the unbroken gauge group in the classical vacuum $|N_1,\ldots,N_d\rangle_\mathrm{cl}$ is given by

$$\mathfrak{h}^{|N_1,\ldots,N_d\rangle_\mathrm{cl}} = \mathfrak{u}(1)^r \oplus \mathfrak{su}(N_{i_1}) \oplus \cdots \oplus \mathfrak{su}(N_{i_r})\,, \tag{2.14}$$

	r=1	2	3	4	5	6	7
N=1	1						
2	2	1					
3	3	4	1				
4	4	10	6	1			
5	5	20	21	8	1		
6	6	35	56	36	10	1	
7	7	56	126	120	55	12	1

Table 1: Values of $\hat{v}_r(N)$ for $1 \leq N \leq 7$ and $1 \leq r \leq N$.

where the r distinct indices i_k correspond to the $N_{i_k} > 0$. Equation (2.8) shows that the quantum vacua can be labeled as $|N_1, k_1; \ldots; N_d, k_d\rangle$ where the integers k_i are defined modulo N_i. The total number of quantum vacua at rank r is thus given by

$$v_r = \binom{d}{r} \hat{v}_r(N) \tag{2.15}$$

with

$$\hat{v}_r(N) = \sum_{\sum_{i=1}^r N_i = N} N_1 \cdots N_r \,. \tag{2.16}$$

It is not difficult to find a generating function for $\hat{v}_r(N)$. If

$$f(x_1, \ldots, x_r) = \prod_{i=1}^r \frac{x_i}{1 - x_i} = \sum_{N_1 \geq 1, \ldots, N_r \geq 1} x_1^{N_1} \cdots x_r^{N_r} \,, \tag{2.17}$$

then

$$g(x) = \frac{\partial^r f}{\partial x_1 \cdots \partial x_r} \bigl(x_1 = x, \ldots, x_r = x\bigr) = \frac{1}{(1-x)^{2r}} = \sum_{N \geq r} \hat{v}_r(N) x^{N-r} \tag{2.18}$$

and this yields

$$\hat{v}_r(N) = \binom{N + r - 1}{2r - 1} \,. \tag{2.19}$$

We list in table 1 the numbers $\hat{v}_r(N)$ for low values of N. These numbers are typically very large, which gives a first indication of the high level of complexity of the model. The case $r = 1$ corresponds to an unbroken gauge group. The N-fold degeneracy, $\hat{v}_1(N) = N$, is similar to what is found in the pure gauge theory. The case $r = N$ corresponds to the Coulomb branch with unbroken gauge group $\mathrm{U}(1)^N$. This branch can be made arbitrarily weakly coupled and there is no chiral symmetry breaking, which explains why $\hat{v}_N(N) = 1$.

Finally, let us note that the number of vacua at rank r (2.15), or the total number of vacua $v = \sum_{r=1}^{\min(d,N)} v_r$, changes only when the degree of the tree-level superpotential changes, which occurs when $g_d = 0$.

Example 4. Our last example is the $\mathrm{U}(N)$ gauge theory with one adjoint chiral superfield $\phi = (\phi^{\mathsf{a}}{}_{\mathsf{b}})$, N_{f} flavours of quarks Q^{a}_f and $\tilde{Q}^f_{\mathsf{a}}$ and tree-level superpotential

$$W_{\text{tree}} = \frac{1}{2}\mu \operatorname{Tr} \phi^2 + \tilde{Q}^f_{\mathsf{a}} m_f^{\ f'}(\phi)^{\mathsf{a}}{}_{\mathsf{b}} Q^{\mathsf{b}}_{f'} \,. \tag{2.20}$$

The matrix-valued polynomial $m_f^{\ f'}(\phi)$ is chosen to be

$$m_f^{\ f'}(\phi) = \delta_f^{f'}(\phi - m_f)\,. \tag{2.21}$$

There is no difficulty in considering more general possibilities, with arbitrary polynomial $m_f^{\ f'}(\phi)$ and a general term $\operatorname{Tr} W(\phi)$ instead of $\frac{1}{2}\mu \operatorname{Tr}\phi^2$ in W_{tree}, but the cases (2.20) and (2.21) are enough to illustrate all the relevant physics of the models (we shall come back on this point in section 4). The classical solutions can be easily obtained by extremizing (2.20). It is found that the eigenvalues of the matrix ϕ can be either equal to zero (which extremizes $W(z) = \frac{1}{2}\mu z^2$) or equal to the m_f. Moreover, at most one eigenvalue of ϕ can be equal to any given m_f. The solutions are thus labeled as $|n; \nu_1, \ldots, \nu_{N_{\mathrm{f}}}\rangle_{\mathrm{cl}}$, with n denoting the number of zero eigenvalues and $\nu_f = 0$ or 1 according to whether there is an eigenvalue equal to m_f or not. Taking into account the constraint $n + \sum_f \nu_f = N$, we find that the total number of classical vacua is given by

$$v_{\mathrm{cl}} = \sum_{k=0}^{\min(N_{\mathrm{f}}, N)} \binom{N_{\mathrm{f}}}{k}\,. \tag{2.22}$$

In particular,

$$v_{\mathrm{cl}} = 2^{N_{\mathrm{f}}} \quad \text{for } N_{\mathrm{f}} \leq N\,. \tag{2.23}$$

In $|n; \nu_1, \ldots, \nu_{N_{\mathrm{f}}}\rangle_{\mathrm{cl}}$, the quarks have non-zero expectation values when some of the ν_f are non-zero and the gauge group is Higgsed down to $\mathrm{U}(n)$. In the quantum theory, there are thus

$$v_1 = \sum_{k=0}^{\min(N_{\mathrm{f}}, N)} (N - k)\binom{N_{\mathrm{f}}}{k} \tag{2.24}$$

rank one vacua, corresponding to $n \geq 1$ and a low energy gauge group $\mathrm{U}(1)$. In particular,

$$v_1 = (2N - N_{\mathrm{f}})2^{N_{\mathrm{f}}-1} \quad \text{for } N_{\mathrm{f}} \leq N\,. \tag{2.25}$$

If $N_{\mathrm{f}} \geq N$, there are also $v_0 = \binom{N_{\mathrm{f}}}{N}$ rank zero vacua in which the gauge group is completely broken.

The model (2.20) is ideal to study the relation between the confining and Higgs regimes as described in section 1.4. Consider for example the case $N_{\mathrm{f}} = N - 1$ (all the other cases display similar phenomena). This model has $n\binom{N-1}{N-n}$ quantum vacua corresponding to classical solutions with unbroken gauge group $\mathrm{U}(n)$, for any $1 \leq n \leq N$. When $n = N$, the gauge group is unbroken and we find the usual N strongly coupled "confining" vacua, similar to the vacua of the pure gauge theory. In particular, classically, the quark fields have zero expectation values in these vacua. On the other hand, when $n = 1$, the gauge group is completely broken (except for the trivial global U(1) factor in U(N)) by the quarks expectation values and we find the weakly coupled "Higgs" vacuum. Intermediate values of n correspond to partially Higgsed vacua. At the classical or semi-classical levels, vacua with different values of n look completely different, and in particular it is impossible to interpolate smoothly between them by varying the parameters. However we shall prove in section 4 that in the full quantum theory the $(N+1)2^{N-2}$ vacua of this model, with all the possible patterns of gauge symmetry breaking $\mathrm{U}(N) \to \mathrm{U}(n)$ for $1 \leq n \leq N$, are actually in the same phase!

2.3 The theory space and monodromies

2.3.1 Global coordinates on theory space

The parameters $\boldsymbol{g}$ and $\boldsymbol{q}$ play a distinguished rôle. For example, the definition of the q_α in (2.4) is motivated by the 2π periodicity in the angles θ_α given in (2.2). The precise statement is as follows.

Proposition 1. *The parameters $(\boldsymbol{g}, \boldsymbol{q})$ are good global coordinates in theory space. In other words, the theory is uniquely defined once we choose $\boldsymbol{g}$ and $\boldsymbol{q}$ and conversely, to a given theory corresponds a unique choice of $\boldsymbol{g}$ and $\boldsymbol{q}$.*

For example, the theories corresponding to the angles θ_α and $\theta_\alpha + 2\pi n_\alpha$, for any integers n_α, must be the same and are associated with the same values of $\boldsymbol{q}$. On the other hand, fractional powers of the instanton factors are not good coordinates since for example $q_\alpha^{1/2}$ and $-q_\alpha^{1/2}$ both correspond to the same theory. Similarly, q_α^2 is not a good coordinate, because two distinct theories, corresponding to q_α and $-q_\alpha$, both have the same q_α^2.

How can we prove proposition 1? In perturbation theory, it is a trivial statement. Beyond perturbation theory, the standard argument is to invoke instantons. Instanton contributions are indeed proportional to some powers of the q_α. However, this argument, in its simplest form, is not correct. The instanton calculus is a semi-classical approximation and thus applies only at weak coupling. On the other hand, proposition 1 is supposed to be valid in all cases, including in theories like the pure gauge theories that have strongly coupled vacua.

Providing a full proof of proposition 1 requires a rigorous, axiomatic definition of the super Yang-Mills theories. This definition doesn't exist for arbitrary correlators, but it does exist in the case of the chiral sector we are interested in [1−3]. The validity of proposition 1 is then a direct consequence of the formalism. We cannot provide the full details here, but the idea is as follows. It turns out that the full information on the chiral sector can be encoded in a microscopic quantum effective superpotential W_{mic} that can always be computed in the instanton approximation for reasons explained in details in [1]. The physics is described by the critical points of the microscopic superpotential. The instanton series for W_{mic} has a finite radius of convergence. The critical points that are located inside the radius of convergence correspond to weakly coupled vacua and the other critical points correspond to strongly coupled vacua. As discussed in the next subsection, in these vacua the expectation values are not 2π periodic in the θ_α, but this is still consistent with proposition 1.[2]

2.3.2 Monodromies amongst the vacua

Let us first consider a weakly coupled vacuum $|i\rangle$ in which the instanton approximation is valid. The analytic function $\mathcal{O}_i(\boldsymbol{g},\boldsymbol{q})$ is then given by a power series in the q_α. In particular, $\mathcal{O}_i(\boldsymbol{g},\boldsymbol{q})$ is 2π periodic in the θ angles,

$$\mathcal{O}_i(\boldsymbol{g}, e^{2i\pi n_\alpha} q_\alpha) = \mathcal{O}_i(\boldsymbol{g},\boldsymbol{q}) \tag{2.26}$$

for any integers n_α.

Proposition 1 allows a more general behaviour than (2.26) and actually the 2π periodicity of the correlators can be violated [20]. To understand the most general possibility, let us start for some values $(\boldsymbol{g},\boldsymbol{q})$ of the parameters and perform an analytic continuation along a closed loop in theory space. Proposition 1 implies that the theory and thus the set of vacua $\{|i\rangle\}$ must be the same before and after the analytic continuation. In other words, if $|i\rangle$ is transformed into $|i\rangle'$ under the analytic continuation, then there must exist a permutation σ such that

$$|i\rangle' = |\sigma(i)\rangle\,. \tag{2.27}$$

Equivalently, the analytic functions $\mathcal{O}_i$ transform as

$$\langle i|\mathcal{O}|i\rangle = \mathcal{O}_i(\boldsymbol{g},\boldsymbol{q}) \longrightarrow \mathcal{O}_{\sigma(i)}(\boldsymbol{g},\boldsymbol{q}) = \langle\sigma(i)|\mathcal{O}|\sigma(i)\rangle\,. \tag{2.28}$$

When strongly coupled vacua are present the permutation σ can be non-trivial.

Performing 2π shifts in the θ angles correspond to particular closed loops in theory space and thus (2.28) implies that in general (2.26) is replaced by

$$\mathcal{O}_i(\boldsymbol{g}, e^{2i\pi n_\alpha} q_\alpha) = \mathcal{O}_{\sigma(i)}(\boldsymbol{g},\boldsymbol{q})\,, \tag{2.29}$$

for some permutation σ that depends on the integers n_α. We see explicitly that vacuum expectation values are not necessarily 2π periodic in the θ angles. In some simple cases, as in the pure gauge theories, the vacua $|i\rangle$ and $|\sigma(i)\rangle$ are related by broken symmetry generators and are thus physically equivalent. However, this is not the case in general: *the physics (i.e. the physical measurements) of the theories is not, in general, 2π periodic in the θ angles.* The meaning of proposition 1 is that the theory must be 2π periodic *as a whole*, when all the vacua are taken into account at the same time. Note that in the special cases where there is only one vacuum, or when all the vacua are related by broken symmetry generators, then the physics is automatically 2π periodic. This is what is believed to happen in non-supersymmetric models.

[2] The 2π periodicity in the θ angles is conjectured to be valid in non-supersymmetric theories as well. A rigorous justification of this fact must await the rigorous construction of the quantum gauge theories. A heuristic argument in favour of 2π periodicity is that the definition of the theory is essentially a UV problem. For asymptotically free gauge theories, the UV is arbitrarily weakly coupled, and thus arguments based on instantons are likely to be correct for this particular purpose (even though they do not give a sensible approximation to the physical correlators). The microscopic construction of the supersymmetric models in [1 – 3] is perfectly consistent with this heuristic idea.

	ϑ	λ	ϕ	ψ	$\mathcal{O}_k$	g_k	q
$\mathrm{U(1)_R}$	$3/2$	$3/2$	1	$-1/2$	δ_k	$3-\delta_k$	$3N-\frac{1}{2}\sum_\chi I_\chi$

Table 2: Charge asignments for the $\mathrm{U(1)_R}$ symmetry of a general gauge theory. The variables ϑ are the superspace coordinates, λ is the gluino, ϕ is the lowest component of an arbitrary scalar chiral superfield, ψ its supersymmetric partner, $\mathcal{O}_k$ a chiral operator in the tree-level superpotential (2.5), g_k the associated coupling, and q an instanton factor. The charge of q is given by the usual chiral anomaly, the sum over χ corresponding to a sum over all the spinor fields coupled to the simple factor of the gauge group associated with q, I_χ being the index of the gauge group representation in which χ transforms.

2.4 The polynomial equations

Proposition 1 can be used to derive a very useful property of the analytic functions $\langle\mathcal{O}\rangle(\boldsymbol{g},\boldsymbol{q})$.

Theorem 2. *For any supersymmetric gauge theory with a finite number v of vacua, there exists a ring a, called the ring of parameters, which is a subring of the ring of entire functions in the parameters $\boldsymbol{g}$ and $\boldsymbol{q}$, such that the expectation value of any chiral operator $\mathcal{O}$ satisfies a degree v polynomial equation with coefficients in a:*

$$P_{\mathcal{O}}(\langle\mathcal{O}\rangle)=0\,,\quad P_{\mathcal{O}}\in\mathsf{a}[X]\,,\ \deg P_{\mathcal{O}}=v\,. \tag{2.30}$$

Moreover, if there exists a U(1) symmetry for which the charges of the fundamental chiral fields and of the parameters $\boldsymbol{g}$ and $\boldsymbol{q}$ are all strictly positive, then $\mathsf{a}=\mathbb{C}[\boldsymbol{g},\boldsymbol{q}]$ is the polynomial ring in the variables $\boldsymbol{g}$ and $\boldsymbol{q}$.

2.4.1 Discussion of the theorem

The non-trivial content of theorem 2 is not in the existence of algebraic equations satisfied by the expectation values (by itself this is an empty statement), but in the fact that *the coefficients of these algebraic equations are contrained to be elements of a particular ring.* In this sense, the analytic functions $\langle\mathcal{O}\rangle$ are similar with respect to the ring a to numbers like $\sqrt{2}$ with respect to the ring of integers $\mathbb{Z}$.

For many purposes the ring a can be replaced by it field of fractions k,[3] that we shall call the *field of parameters.* One interest in using k instead of a is that the polynomials in theorem 2 can be constrained to be monic, i.e. of the form $P_{\mathcal{O}}(X)=X^v+\cdots$ For example, if $\mathsf{a}=\mathbb{C}[\boldsymbol{g},\boldsymbol{q}]$, then $\mathsf{k}=\mathbb{C}(\boldsymbol{g},\boldsymbol{q})$ is the field of rational functions in the parameters $\boldsymbol{g}$ and $\boldsymbol{q}$. In this case, an equation with coefficients in k actually automatically yields an equation with coefficients in a, since we can always clear the denominators of the coefficients by multiplying by their least common multiple.

In the following, the reader may always assume that $\mathsf{a}=\mathbb{C}[\boldsymbol{g},\boldsymbol{q}]$ is the polynomial ring. The assumption in theorem 2 that ensures that this is the case is a relatively minor technical requirement satisfied in a lot of models. For example, all the super Yang-Mills theories have

[3]The field of fraction exists because a, being a subring of the ring of entire functions, is an integral domain.

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a $\mathrm{U}(1)_{\mathrm{R}}$ symmetry defined by identifying the $\mathrm{U}(1)_{\mathrm{R}}$ charges with the canonical dimensions of the chiral superfields, see table 2. This symmetry satisfies the conditions of the theorem provided the model is asymptotically free (which yields a positive charge for q) and the tree-level superpotential includes only super-renormalizable terms (which corresponds to positive charges for the g_k). For instance, example 2 in section 2.2.2 is of this type. Renormalizable (but not super-renormalizable) terms, associated with couplings of zero $\mathrm{U}(1)_{\mathrm{R}}$ charge, can also be included in many cases, because the renormalizable couplings can often be absorbed in suitable field redefinitions (in other words, the dependence in these couplings can be straightforwardly derived by simple rescalings). For instance, this is what we have done in example 4 by choosing the leading term in (2.21) to be $\delta^f_{f'}$ instead of $g\delta^f_{f'}$ for an arbitrary coupling g. Even models including non-renormalizable terms[4] often satisfy the assumption in the theorem. For instance, example 3 does have non-renormalizable couplings in the tree-level superpotential (2.10) when $\deg W > 3$. However, the model has another R-symmetry $\mathrm{U}(1)'_{\mathrm{R}}$ with charge asignments

$$\begin{array}{c|cccccc} & \vartheta & \lambda & \phi & \psi & g_k & q \\ \mathrm{U}(1)'_{\mathrm{R}} & 1 & 1 & 0 & -1 & 2 & 0 \end{array}\,. \tag{2.31}$$

It is always possible to find a linear combination of $\mathrm{U}(1)_{\mathrm{R}}$ and $\mathrm{U}(1)'_{\mathrm{R}}$ that satisfies the conditions of the theorem.

On the other hand, in theories with zero β functions, the ring a can include arbitrary power series in the instanton factors. For example, if the theory has a S-duality, the coefficients of the polynomials of theorem 2 typically involve modular forms.

2.4.2 Proof of the theorem

Let $\mathcal{O}$ be a chiral operator, $\mathcal{O}_i = \langle i|\mathcal{O}|i\rangle$ and consider the monic polynomial

$$\hat{P}_{\mathcal{O}}(X) = \prod_{i=1}^{v}\bigl(X - \mathcal{O}_i(\boldsymbol{g},\boldsymbol{q})\bigr) = X^v + \sum_{k=1}^{v}\hat{a}_k(\boldsymbol{g},\boldsymbol{q})X^{v-k}\,. \tag{2.32}$$

By construction, $\hat{P}_{\mathcal{O}}(\langle\mathcal{O}\rangle) = 0$.

Let us perform an analytic continuation along an arbitrary closed loop in the space of parameters $(\boldsymbol{g},\boldsymbol{q})$. From (2.28), we find that

$$\hat{P}_{\mathcal{O}}(X) \longrightarrow \prod_{i=1}^{v}\bigl(X - \mathcal{O}_{\sigma(i)}(\boldsymbol{g},\boldsymbol{q})\bigr) = \prod_{i=1}^{v}\bigl(X - \mathcal{O}_i(\boldsymbol{g},\boldsymbol{q})\bigr) = \hat{P}_{\mathcal{O}}(X)\,, \tag{2.33}$$

and thus the coefficients $\hat{a}_k(\boldsymbol{g},\boldsymbol{q})$ defined in (2.32) are *single*-valued analytic functions of $\boldsymbol{g}$ and $\boldsymbol{q}$.

[4]These terms occur in a string theory context where the field theory is viewed as a low energy approximation and yield interesting physics. Even from a purely field theoretic point of view it is perfectly consistent to include them when one focuses on the chiral sector of the theory. This is so because the necessary counterterms are governed by the UV cut-off which is a real parameter and thus does not affect the chiral sector.

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Singularities of the functions $\hat{a}_k$ can only occur when some of the vacua disappear from the spectrum. From the discussion of section 2.2.2, we know that the positions of the singularities can thus be derived by a purely classical analysis. Moreover, near a singular point, the effective superpotential evaluated in the vacuum that disappears at the singularity is arbitrarily large. The leading singular behaviour of the expectation values can then be obtained from a classical analysis as well. For tree-level superpotentials of the form (2.5), this is always given by a power-law divergence. The conclusion is that the $\hat{a}_k(\boldsymbol{g}, \boldsymbol{q})$ are meromorphic functions with a finite number of poles. Multiplying $\hat{P}_{\mathcal{O}}$ by a suitable polynomial in the parameters $\boldsymbol{g}$ and $\boldsymbol{q}$ to clear up these poles, we obtain the polynomial $P_{\mathcal{O}}$ of theorem 2.

Let us now assume that there exists a U(1) symmetry for which the charges of the fundamental chiral fields and of the parameters $\boldsymbol{g}$ and $\boldsymbol{q}$ are all strictly positive. An arbitrary chiral operator $\mathcal{O}$ can be written as a sum of operators of strictly positive U(1) charges. Let $\delta > 0$ be the greatest of these charges. If we asign to the dummy variable X in (2.32) the charge δ, then $\hat{P}_{\mathcal{O}}$ is a sum of terms whose charges are bounded by $v\delta$. The polynomial $P_{\mathcal{O}}$, which is obtained from $\hat{P}_{\mathcal{O}}$ by multiplying by a polynomial in $\boldsymbol{g}$ and $\boldsymbol{q}$, is thus also a sum of terms of given U(1) charges, these charges being bounded by a certain strictly positive integer. Let us write

$$P_{\mathcal{O}}(X) = \sum_{k=0}^{v} a_k(\boldsymbol{g}, \boldsymbol{q}) X^{v-k} \, . \tag{2.34}$$

The a_k are entire functions and can thus be expanded as power series in $\boldsymbol{g}$ and $\boldsymbol{q}$. From the above discussion, they have a maximum U(1) charge. Since the variables $\boldsymbol{g}$ and $\boldsymbol{q}$ have strictly positive U(1) charges, the power series must terminate after a finite number of terms, and thus $a_k \in \mathbb{C}[\boldsymbol{g}, \boldsymbol{q}]$.

2.4.3 The power of the polynomial equations

Proposition 3. *The full solution of the model, i.e. the full set of expectation values* $\langle i|\mathcal{O}|i\rangle$ *for all chiral operators* $\mathcal{O}$ *and all the vacua* $|i\rangle$*,* $1 \le i \le v$*, can be derived from the knowledge of a finite number of the polynomial equations of theorem 2.*

Let us explain the significance of this result. If one picks a given operator $\mathcal{O}$, then by construction there are v solutions to the polynomial equation $P_{\mathcal{O}} = 0$, corresponding to the v expectation values $\langle i|\mathcal{O}|i\rangle$. Which solution corresponds to which vacuum is a matter of convention and we can always choose to label the vacua according to a particular labeling of the roots of $P_{\mathcal{O}}$. Let us assume that we have chosen a particular labeling. Let us now consider another operator $\mathcal{O}'$. One can find the *unordered* set of v expectation values of $\mathcal{O}'$ by solving $P_{\mathcal{O}'} = 0$. However, we do not know which root corresponds to which vacuum. This is no longer a matter of arbitrary choice, since the vacua have already been labeled. So we see that the knowledge of $P_{\mathcal{O}}$ and $P_{\mathcal{O}'}$ is not enough to derive the expectation values of $\mathcal{O}$ and $\mathcal{O}'$, there remains an ambiguity corresponding to the permutation of the vacua. Of course, additional constraints can be found by considering more polynomials, like $P_{\mathcal{O}\mathcal{O}'}$ for example. Proposition 3 states that all the ambiguity, for all the expectation values, can be cleared up by considering a finite set of equations of the form (2.30). The proof of this result will be given in 2.5.6 after more technical tools are introduced.

2.4.4 Simple examples

Example 5. In the case of the pure gauge theory based on a simple gauge group G, the most general chiral operator is a polynomial in the glueball operator S, defined in terms of the super field strength W^α as

$$S = -\frac{1}{16\pi^2}\,\mathrm{Tr}\, W^\alpha W_\alpha\,. \tag{2.35}$$

The expectation value of this operator satisfies the equation

$$\langle S\rangle^{h^\vee} = q\,, \tag{2.36}$$

where $h^\vee$ is the dual Coxeter number of G. Thus the polynomial for S is simply

$$P_S(X) = X^{h^\vee} - q \in \mathbb{C}[q][X]\,. \tag{2.37}$$

Example 6. In the model (2.9), the polynomial for the glueball operator is simply $P_S(X) = X^N - q\det m$. The mesonic operator $M_{f'}^{\ f} = \tilde{Q}^f Q_{f'}$ expectation values also satisfy degree N algebraic equations with coefficients in $\mathbb{C}[q, m_f^{\ f'}]$ that straightforwardly follow from the relation $\langle M_{f'}^{\ f}\rangle = (m^{-1})_{f'}^{\ f}\langle S\rangle$.

2.4.5 A clarifying remark

Let us here stress a point that has been at the origin of some considerable confusion in the literature. The fact that the coefficients of the polynomials $P_{\mathcal{O}}$ are polynomials in the instanton factors (and not, for example, in arbitrary fractional powers of these factors), might lead one to believe that the result relies on some semi-classical instanton analysis. *This is not true.* The arguments that we have used to derive the result are valid in the full strongly coupled quantum theory. The fact that only integer powers of q enter in the coefficients of $P_{\mathcal{O}}$ comes from an argument based on analyticity and *not* from an argument based on a weakly coupled approximation. In particular, *the coefficients of the polynomials $P_{\mathcal{O}}$ cannot be computed in general from a straightforward instanton calculation.* Another facet of this subtlety is that the expectation values, which are the solutions of the polynomial equations $P_{\mathcal{O}} = 0$, usually do not have expansions in integer powers of q.

For example, the fact that only q enters the equation (2.36) suggested in the old literature that the relation could be derived by a direct instanton calculation in the pure gauge theory and this yielded some inconsistencies. This is not surprising. The pure gauge theory is strongly coupled and (2.36) *cannot* be derived by a direct semi-classical calculation in this theory.

2.5 The chiral ring

We are now ready to define the fundamental notion of the quantum chiral ring. This concept is well-known, but a precise definition in the non-perturbative quantum theory

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does not seem to have appeared previously in the literature. Much more importantly, the full significance of this notion has not been fully appreciated and its power was used only very recently [21]. Understanding the structure of the chiral ring will give us the keys to understanding the phase structure of the models.

Our definition in section 2.5.2 of the quantum chiral ring is motivated by the following two fundamental properties.

Proposition 4. *(i) The full solution of the theory in the chiral sector is coded in the chiral ring* A*, i.e. we can compute the analytic functions* $\langle \mathcal{O} \rangle$*, for all the chiral operators* $\mathcal{O}$*, from the knowledge of the ring* A*. (ii) The chiral ring contains only physical information.*

Presenting the solution of a gauge theory via an algebraic structure like a ring may be unfamiliar. The main interest in doing so is that the ring A does not contain any unphysical, "scheme-dependent" information. On the other hand, and as will become clear in the examples below, the usual ways of presenting the solutions, for example using effective superpotentials or generating functions for expectation values, do contain a lot of unphysical information that can obscure the physics.

2.5.1 On the classical chiral ring

Let us start by reviewing the simple notion of the classical chiral ring. The construction starts by building all the chiral operators by forming appropriate gauge invariant polynomials in the elementary chiral fields. A very important property, that follows immediately from the fact that a field theory has only a finite number of elementary fields, is that the most general chiral operator $\mathcal{O}$ can be written as a polynomial $\rho_{\mathcal{O}}$ in a *finite* number of generators,

$$\mathcal{O} = \rho_{\mathcal{O}}(\mathcal{O}_1, \ldots, \mathcal{O}_m) \,. \tag{2.38}$$

The generators satisfy algebraic identities that come from their definitions in terms of the gauge-variant elementary fields (these identities are called sygyzies). Moreover, there are relations that follow from the extremization of the tree-level superpotential (the so-called F-term conditions). A standard definition of the classical chiral ring is then given by considering only the bosonic generators $\mathcal{O}_1, \ldots, \mathcal{O}_n$ and by taking the quotient of the polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$ in n variables with the ideal I generated by the set of all the above-mentioned relations,

$$\mathsf{A}_{\mathrm{cl,\,standard}} = \mathbb{C}[X_1, \ldots, X_n]/I \,. \tag{2.39}$$

Sometimes, one considers only generators built from bosonic elementary superfields, thus excluding fermion bilinears for examples.[5] In spite of the fact that $\mathsf{A}_{\mathrm{cl,\,standard}}$ is a purely classical object, it can have a rather complex and interesting structure. For example, in the case of theories that are built in string theory by putting D-branes at Calabi-Yau singularities, the classical chiral ring encodes in a very interesting way the Calabi-Yau geometry and many additional useful informations [11].

[5]Discarding fermionic variables is justified at the classical level since fields build from them will automatically have zero classical expectation values.

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In the standard approach, the polynomials in (2.38) have coefficients in the field of complex numbers and the parameters $\boldsymbol{g}$ of the classical theory are simply considered to be complex numbers as well. This point of view is insufficient for our purposes, which is to be able to reconstruct all the expectation values (for the moment in the classical theory) as functions of $\boldsymbol{g}$ from the structure of the chiral ring only. To do that, one must consider the parameters $\boldsymbol{g}$ to be "dummy variables," or in other words to include them as new generators in the chiral ring. In this point of view, the polynomials in (2.38) will be elements of the polynomial ring $\mathbb{C}[\boldsymbol{g}][X_1,\dots,X_n]$, and we define

$$\mathsf{A}_{\mathrm{cl}} = \mathbb{C}[\boldsymbol{g}][X_1,\dots,X_n]/I\,, \tag{2.40}$$

where I is now the ideal generated by all the relations between the generators that are polynomials with coefficients in $\mathbb{C}[\boldsymbol{g}]$. This definition is sensible because it turns out that all the syzygies and all the F-term constraints are equivalent to polynomial constraints *with coefficients in* $\mathbb{C}[\boldsymbol{g}]$. This is a crucial point, that we are going to develop further in the general case of the non-perturbative quantum theory.

2.5.2 The definition of the quantum chiral ring

To define the chiral ring at the quantum level, with the properties listed in proposition 4 in mind, we cannot, as we have just done in the classical context, refer to the gauge-variant elementary fields of the theory. Indeed, the physical content of the theory is entirely coded in the gauge invariant variables. In particular, at the quantum level, we want to be able to describe situations where different classical theories, with different gauge-variant elementary fields and/or gauge groups, can yield physically equivalent "dual" quantum theories.

The only data that we must borrow from the classical theory is a list of chiral operators $\mathcal{O}_1,\dots,\mathcal{O}_n$ that form a set of generators for all the chiral operators of the theory (note that the identity operator is always present and in general we do not include it explicitly in the list of generators). This can be seen as a basic axiom of what we mean by quantizing a given classical theory. If a is the ring of parameters (see theorem 2), we define the most general chiral operator of the theory to be any finite sum of finite products of the generators $\mathcal{O}_i$ with coefficients in a,

$$\mathcal{O} = \rho_{\mathcal{O}}(\mathcal{O}_1,\dots,\mathcal{O}_n)\,,\quad \rho_{\mathcal{O}} \in \mathsf{a}[X_1,\dots,X_n]\,. \tag{2.41}$$

Definition 1. Let $\mathcal{O}^{(1)},\dots,\mathcal{O}^{(p)}$ be p chiral operators, i.e. operators of the form (2.41). An *operator relation* between the $\mathcal{O}^{(i)}$ is a polynomial equation of the form

$$P(\mathcal{O}^{(1)},\dots,\mathcal{O}^{(p)}) = 0\,,\quad P \in \mathsf{a}[X_1,\dots,X_p]\,, \tag{2.42}$$

such that $P(\langle i|\mathcal{O}^{(1)}|i\rangle,\dots,\langle i|\mathcal{O}^{(p)}|i\rangle)$ identically vanishes in all the vacua $|i\rangle$ of the theory.

Note that this definition is unambiguous because of the well-known factorization of chiral operators expectation values,

$$\langle \mathcal{O}\mathcal{O}'\rangle = \langle\mathcal{O}\rangle\langle\mathcal{O}'\rangle\,, \tag{2.43}$$

which follows from the space-time independence of the chiral correlators and from the cluster decomposition principle. In particular,

$$P\big(\langle i|\mathcal{O}^{(1)}|i\rangle,\dots,\langle i|\mathcal{O}^{(p)}|i\rangle\big) = \big\langle i\big|P(\mathcal{O}^{(1)},\dots,\mathcal{O}^{(p)})\big|i\big\rangle\,. \tag{2.44}$$

An operator relation in the sense of definition 1 thus has two basic properties: first it is a relation valid in all the vacua of the theory; second it is a polynomial relation with coefficients in a.

Definition 2. The quantum chiral ring $\mathsf{A}=\mathsf{a}[\mathcal{O}_1,\dots,\mathcal{O}_n]$ is the ring of all chiral operators of the form (2.41), taking into account all the operator relations of the form (2.42). In other words, there is a canonical surjective ring homomorphism from the polynomial ring $\mathsf{a}[X_1,\dots,X_n]$ onto A obtained by mapping X_i to $\mathcal{O}_i$. The kernel of this mapping is the ideal $\mathscr{I}$ generated by all the operator relations and A is isomorphic to the ring quotient

$$\mathsf{A}=\mathsf{a}[X_1,\dots,X_n]/\mathscr{I}\,. \tag{2.45}$$

2.5.3 The perturbative chiral ring

It is useful to define the notion of a perturbative chiral ring A_{pert}. The motivation behind this concept is to make precise the notion of quantum corrections: the quantum corrections are non-trivial is A and A_{pert} are not isomorphic and are trivial otherwise.

In perturbation theory, the standard non-renormalization theorem ensures that the chiral operators expectation values are not quantum corrected. This motivates the following definition.

Definition 3. The perturbative chiral ring A_{pert} is defined as the quantum chiral ring in definition 2, except that we set to zero all the instanton factors in the quantum operator relations,

$$\mathsf{A}_{\text{pert}}=\mathsf{A}/(\boldsymbol{q})[\boldsymbol{q}]\,. \tag{2.46}$$

In many (but not necessarily all) cases, the perturbative chiral ring simply coincides with the classical chiral ring defined in (2.40), except that the variables $\boldsymbol{q}$ are added,

$$\mathsf{A}_{\text{pert}}=\mathsf{A}_{\text{cl}}[\boldsymbol{q}]\,. \tag{2.47}$$

2.5.4 Simple algebraic properties of the quantum chiral ring

The rings A in definition 2 are not generic rings but have some special properties that we now discuss.

The ring A is commutative. In general, gauge invariant chiral operators can include both bosonic and fermionic operators. However, fermionic operators automatically have zero expectation values in a Lorentz-invariant theory. Our definition of the chiral ring then implies that only the bosonic operators need to be taken into account and thus A is always commutative. Let us note that this requirement could be lifted by introducing Lorentz-violating couplings to the fermionic chiral operators in the tree-level superpotential. It is straightforward to develop a generalized theory that includes these terms but, since we are not aware of any useful physical application of such a construction, we shall restrict ourselves to Lorentz invariant theories.

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The ring A has no nilpotent element. A nilpotent element x is a non-zero element such that $x^r = 0$ for some integer $r > 1$. However, $x^r = 0$ in A means that $\langle x^r \rangle = \langle x \rangle^r = 0$ in all the vacua of the theory. This in turn implies that $\langle x \rangle = 0$ in all the vacua and thus that $x = 0$ in A.

The fact that A has no nilpotent element can be expressed in terms of the ideal $\mathscr{I}$ of operator relations. In general, for any ideal I of a commutative ring A, one defines the radical $r(I)$ of I to be the set of elements x of A such that $x^r \in I$ for some $r \geq 1$,

$$r(I) = \{x \in A \mid \exists\, r > 0\,,\ x^r \in I\}\,. \tag{2.48}$$

It is straightforward to check that $r(I)$ is itself an ideal, that $r(r(I)) = r(I)$ and that A/I has no nilpotent element if and only if $r(I) = I$ in which case we say that I is a radical ideal. Thus the ideal $\mathscr{I}$ of operator relations is radical.

Let us note that the classical or perturbative rings as defined in 2.5.1 and 2.5.3 can have nilpotent elements. Thus *perturbative (or classical) chiral rings are not special cases of quantum chiral rings.* This exhibits the singular nature of the classical limit and will be illustrated in example 7 below.

A finite dimensional vector space. As we have already briefly discussed in section 2.4.1, it is natural for many purposes to enlarge the set of chiral operators by allowing the coefficients of the polynomials in (2.41) to be elements of the field of fractions $\mathsf{k} = \mathrm{Frac}(\mathsf{a})$ instead of a. The enlarged chiral ring $\mathbb{A}$ will be simply defined by

$$\mathbb{A} = \mathsf{k}[\mathcal{O}_1, \ldots, \mathcal{O}_n] = \mathsf{k}[X_1, \ldots, X_n]/\mathscr{I}\,, \tag{2.49}$$

to be compared with (2.45). Considering the ring $\mathbb{A}$ instead of A doesn't change the physics but can help to simplify the mathematics. For example, it is clear that the ring $\mathbb{A}$ is a k vector space. Interestingly, it is a finite dimensional vector space. In particular, if $(\mathcal{B}_\alpha)_{1 \leq \alpha \leq \dim_{\mathsf{k}} \mathbb{A}}$ is a basis, then any chiral operator $\mathcal{O} \in \mathbb{A}$ can be expanded as

$$\mathcal{O} = \sum_{\alpha=1}^{\dim_{\mathsf{k}} \mathbb{A}} c_\alpha \mathcal{B}_\alpha\,, \tag{2.50}$$

where $c_\alpha \in \mathsf{k}$.

The relation (2.50) is interesting because it is *linear*, unlike the non-linear relations of the form (2.41). The proof of the existence of a finite basis $(\mathcal{B}_\alpha)$ relies on theorem 2. For example, assume that the ring is generated by only one operator $\mathcal{O}_1$. By using the polynomial equation satisfied by $\mathcal{O}_1$ one can express $\mathcal{O}_1^p$, for any $p \geq v$, as a linear combination of $(\mathbb{I}, \mathcal{O}_1, \ldots, \mathcal{O}_1^{v-1})$. This implies that $\dim_{\mathsf{k}} \mathbb{A} \leq v$. In the general case, the proof can be easily done by induction on the number n of generators.

The ring A is graded. Each $\mathrm{U}(1)$ global symmetry of the gauge theory induces a grading

$$\mathsf{A} = \bigoplus_n \mathsf{A}_n \tag{2.51}$$

where A_n is the set of ring elements having charge n. The important property is that $\mathsf{A}_n \mathsf{A}_m \subset \mathsf{A}_{n+m}$. Note that only A_0 is a subring. The grading implies that the ideal $\mathscr{I}$ is generated by a set of homogeneous polynomials, i.e. by polynomials of fixed $\mathrm{U}(1)$ charges.

The ideal $\mathscr{I}$ is finitely generated. The ideals of the ring $\mathsf{k}[X_1, \ldots, X_n]$ are always finitely generated (one says that the polynomial ring is noetherian). This result applied to the ideal $\mathscr{I}$ implies that there always exists a *finite* number of polynomials $R_i \in \mathsf{k}[X_1, \ldots, X_n]$, $1 \leq i \leq r$, such that any operator relation can be written in the form

$$\sum_{i=1}^{r} a_i R_i = 0 \tag{2.52}$$

with $a_i \in \mathsf{k}[X_1, \ldots, X_n]$. In other words, all the information about the ring $\mathbb{A}$ is encoded in a *finite* number of relations $R_1 = \cdots = R_r = 0$. The same is true for the ring A.[6]

Example 7. To illustrate the above properties, consider the simplest case of the pure $\mathrm{SU}(N)$ gauge theory. The ring of parameters is simply $\mathsf{a} = \mathbb{C}[q]$. The chiral ring is generated by the single operator S defined in (2.35). From (2.36), we deduce that it satisfies the operator relation

$$S^N - q = 0\,. \tag{2.53}$$

Hence $\mathscr{I} = (S^N - q)$ and

$$\mathsf{A} = \mathbb{C}[q, S]/(S^N - q)\,. \tag{2.54}$$

Taking into account (2.53), we see that the most general chiral operator can be written in the form

$$\mathcal{O} = \sum_{k=0}^{N-1} a_k(q) S^k\,, \tag{2.55}$$

where $a_k \in \mathbb{C}[q]$ (if $\mathcal{O} \in \mathsf{A}$) or $a_k \in \mathbb{C}(q)$ (if $\mathcal{O} \in \mathbb{A}$). Clearly, $(1, S, \ldots, S^{N-1})$ is a base of $\mathbb{A}$ over $\mathbb{C}(q)$ and in particular $\dim_{\mathbb{C}(q)} \mathbb{A} = N$. The ring A is graded with respect to a $\mathrm{U}(1)$ symmetry under which S has charge 1 and q has charge N (up to a rescaling of charges, this is the $\mathrm{U}(1)_{\mathrm{R}}$ symmetry described in section 2.4.1).

The perturbative chiral ring is obtained by setting $q = 0$ in (2.53),

$$\mathsf{A}_{\text{pert}} = \mathbb{C}[q, S]/(S^N)\,. \tag{2.56}$$

We see that S is nilpotent in A_{pert}.

2.5.5 Physical properties of the chiral ring

In this subsection, we are going to discuss the fundamental proposition 4.

Property (ii) in the proposition is trivially satisfied, because our definition of the chiral ring relies exclusively on the knowledge of the expectation values $\langle \mathcal{O} \rangle$.

Property (i) means that one can reconstruct in principle all the chiral operators expectation values from the ring A. The procedure to do so is as follows. One first considers the canonical surjection $\mathsf{a}[X_1, \ldots, X_n] \rightarrow \mathsf{A} = \mathsf{a}[\mathcal{O}_1, \ldots, \mathcal{O}_n]$ that maps the dummy unconstrained variables X_i to the operator $\mathcal{O}_i$. The kernel of this mapping is the radical ideal

[6] When a is not noetherian this is a consequence of proposition 3 in 2.4.3.

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$\mathscr{I}$. We then find a set of generators for the ideal, $\mathscr{I} = (R_1, \ldots, R_r)$, which yields a set of algebraic equations for the expectation values,

$$R_i\big(\langle \mathcal{O}_1\rangle, \ldots, \langle \mathcal{O}_n\rangle\big) = 0\,, \quad R_i \in \mathsf{a}[X_1, \ldots, X_n]\,,\ 1 \leq i \leq r\,. \tag{2.57}$$

The question is: do the algebraic equations (2.57), which are constrained to be with coefficients in a, determine unambiguously the analytic functions $\langle \mathcal{O}_i\rangle(\boldsymbol{g}, \boldsymbol{q})$ in all the vacua of the theory?

Before we provide a proof, let us illustrate the result in the case of the pure $\mathrm{SU}(N)$ gauge theory. As explained in the previous subsection (example 7), the ideal $\mathscr{I}$ is in this case generated by the polynomial $S^N - q$, which yields the algebraic equation

$$\langle S\rangle^N - q = 0\,. \tag{2.58}$$

This equation has N solutions associated with the N vacua of the theory,

$$\langle k|S|k\rangle = q^{1/N} e^{2i\pi k/N}\,, \tag{2.59}$$

and this yields indeed the full solution of the model. Let us emphasize that this result strongly depends on the precise definition of A and in particular of the ring a. For example, if instead of $\mathsf{a} = \mathbb{C}[q]$ we had used $\mathbb{C}[q^2]$, then the only relation that could be considered would be $S^{2N} = q^2$, and this has unphysical solutions.

The fundamental ingredient in proving that the algebraic equations with coefficients in a (2.57) give enough information to determine the expectation values is of course the existence of the polynomial equations described in section 2.4. Theorem 2 implies that for any chiral operator $\mathcal{O}$, there exists a degree v polynomial $P_{\mathcal{O}} \in \mathsf{a}[X]$ such that

$$P_{\mathcal{O}}(\mathcal{O}) = 0 \tag{2.60}$$

is an operator relation of the form (2.42). These polynomials (or more precisely the polynomials $P_{\mathcal{O}} \circ \rho_{\mathcal{O}}$ obtained after expressing $\mathcal{O}$ in terms of the generators $\mathcal{O}_1, \ldots, \mathcal{O}_n$ as in (2.41)) are thus automatically in the ideal $\mathscr{I}$, i.e. are linear combinations with coefficients in $\mathsf{a}[X_1, \ldots, X_n]$ of the polynomials R_i appearing in (2.57). In particular, from the equations (2.57) one can derive the condition

$$P_{\mathcal{O}}(\langle \mathcal{O}\rangle) = 0\,. \tag{2.61}$$

Thus all we need to do is to prove the proposition 3 of section 2.4.3.

Remark. Assume that one has a set of polynomials $P_a \in \mathsf{a}[X_1, \ldots, X_n]$ such that the equations $P_a = 0$ determine completely all the chiral operators expectation values. The equations $P_a = 0$ thus encode all the physical information about the theory. Let $I = (P_a)$ be the ideal generated by the polynomials P_a. Then, in general, I is not equal to $\mathscr{I}$ and the quotient ring $\mathsf{a}[X_1, \ldots, X_n]/I$ is not equal to the chiral ring. All that can be said is that $I \subset \mathscr{I}$, however I does not need to be a radical ideal. Physically speaking, this means that I and $\mathsf{a}[X_1, \ldots, X_n]/I$ contain in general unphysical information depending on arbitrary choices. On the other hand, one always has $r(I) = \mathscr{I}$ as a consequence of Hilbert's nullstellansatz, and thus the chiral ring is obtained from $\mathsf{a}[X_1,\ldots,X_n]/I$ by setting to zero all the nilpotent elements.

For example, in the case of the pure $\mathrm{SU}(N)$ gauge theory, one could replace the algebraic equation (2.58) by $(S^N-q)^2=0$. Clearly, the ideal $((S^N-q)^2)$ is strictly included in $\mathscr{I}=(S^N-q)$, and the associated ring $\mathbb{C}[q,S]/((S^N-q)^2)$ has a nilpotent element.

2.5.6 The power of the polynomial equations, again

Strictly speaking, to prove proposition 4 we actually don't need the full power of proposition 3, but only the fact that the full set of polynomial equations (which is infinite) determines unambiguously all the expectation values. So let us start by analysing this weaker statement.

The idea is to consider the operator $\mathcal{O}_{z_1,\ldots,z_n}$ defined by

$$\mathcal{O}_{z_1,\ldots,z_n}=\sum_{\alpha=1}^{n} z_\alpha\mathcal{O}_\alpha\,, \tag{2.62}$$

where the $\mathcal{O}_1,\ldots,\mathcal{O}_n$ form a set of generators of A. The z_α in (2.62) are arbitrary complex numbers. From the polynomial $P_{\mathcal{O}_{z_1,\ldots,z_n}}\in\mathsf{a}[X]$, we can derive the expectation values $\langle i|O_{z_1,\ldots,z_n}|i\rangle$ in all the vacua $|i\rangle$. The important point is that, by continuity in the z_α, there is no ambiguity in labeling the vacua for different values of the z_α. One can then deduce the expectation values of all the generators from

$$\langle i|\mathcal{O}_\alpha|i\rangle=\frac{\partial\langle i|O_{z_1,\ldots,z_n}|i\rangle}{\partial z_\alpha}\,\cdot \tag{2.63}$$

Since the most general chiral operator is of the form (2.41), its expectation value is straightforwardly obtained from (2.63) as well.

To complete the proof of proposition 3, we need to show that actually only a finite number of polynomial equations is needed (in the above argument, we used an infinite set of such equations, labeled by the variables z_α). This follows immediately from the fact that polynomial rings are noetherian, which can be summarized in the following lemma.

Lemma 5. *Let $\mathscr{P}=(P_a)_{a\in\mathscr{A}}$ be an arbitrary family of polynomials in $\mathsf{k}[X_1,\ldots,X_n]$. Then there always exists a finite number of polynomials P_i, $1\leq i\leq p$, $P_i\in\mathscr{P}$, such that any $P\in\mathscr{P}$ can be written as $P=\sum_{i=1}^{p}a_iP_i$ for some $a_i\in\mathsf{k}[X_1,\ldots,X_n]$.*

We refer the reader to standard textbooks [17] for a proof. In our case, the family of polynomials that we consider is formed by all the polynomials of the form $P_{\mathcal{O}}\circ\rho_{\mathcal{O}}$, for all the chiral operators $\mathcal{O}$, with $\rho_{\mathcal{O}}$ defined by (2.41).

2.6 The chiral ring and operator mixing

In this subsection, we are going to illustrate, using very simple examples, the fact that all the physics of the theory in encoded in the chiral ring A and that any additional piece of information must be unphysical (i.e. corresponds to arbitrary choices). All we say is very elementary, yet it clarifies many confusions and correct some errors that are

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commonly found in the literature. As we shall see, an important source of confusion comes from the possibility to define in different ways some composite operators. This ambiguity is a non-perturbative version of the ambiguity associated to a choice of scheme in ordinary perturbative quantum field theory. It is directly related to the freedom one has in performing field redefinitions. Field redefinitions do not change the physics nor the chiral ring, but they can drastically change the way the solution of the model is presented.

Example 8. Let us start by considering once more the case of the pure gauge theory, but this time with gauge group $\mathrm{U}(N)$ instead of $\mathrm{SU}(N)$. This yields the following puzzle.[7] The solution of the model is still given by (2.59), which is often summarized by saying that the effective quantum glueball superpotential is given by the Veneziano-Yankielowicz formula,

$$W(S) = -S \ln \frac{S^N}{e^N q} \cdot \tag{2.64}$$

It is indeed straightforward to check that the equations $W'(S) = 0$ yield the solutions (2.59). Let us now consider the case $N = 1$. On the one hand, since the gauge theory is in this case a free $\mathrm{U}(1)$ theory, we do not expect any non-trivial quantum correction. However, we still have a non-trivial glueball superpotential (2.64) and a non-trivial gluino condensate

$$\langle S \rangle = q\,. \tag{2.65}$$

How is this possible?

One interpretation, advocated in [22], is that to any classical super Yang-Mills theory is associated an infinite number of physically inequivalent quantum theories with the same classical limit. The $\mathrm{U}(1)$ theory with the condensate (2.65) would then correspond to a non-standard way to quantize the abelian gauge theory (or to a non-standard UV completion in the language of [22]), which would yield a non-trivial quantum abelian gauge theory.

We do not subscribe to this interpretation. Actually, we shall make clear that there is always a unique quantum supersymmetric gauge theory associated with a given classical supersymmetric gauge theory and that the ambiguities described in [22] correspond to field or parameter redefinitions.

To understand how this works for our simple $\mathrm{U}(1)$ example, let us compute the chiral ring. At the perturbative level, the $\mathrm{U}(1)$ theory has no non-trivial chiral operator except of course the identity $\mathbb{I}$ and the perturbative chiral ring is given by

$$\mathsf{A}_{\mathrm{pert}} = \mathbb{C}[q]\,. \tag{2.66}$$

On the other hand, the quantum chiral ring associated with (2.65) is given by (this is simply (2.54) for $N = 1$)

$$\mathsf{A} = \mathbb{C}[q, S]/(S - q)\,. \tag{2.67}$$

The rings $\mathsf{A}_{\mathrm{pert}}$ and A are clearly isomorphic,

$$\mathsf{A} = \mathsf{A}_{\mathrm{pert}}\,. \tag{2.68}$$

[7] I would like to thank Mina Aganagic for bringing this puzzle to my attention.

From the discussion in previous sections, we know that this implies that the U(1) theory does not have any non-trivial quantum corrections. In particular, the result (2.65) and the glueball superpotential (2.64) for $N = 1$ are completely unphysical.

These statements might still appear surprising, so let us spell their meaning in a very concrete way. When one makes the claim that the U(1) theory has a non-trivial condensate (2.65), one actually has forgotten to analyse precisely the definition of the operator S in the quantum theory. As we shall explain in details below, the operator S (as many other commonly used operators in supersymmetric gauge theories) is ambiguous in the non-perturbative U(1) quantum theory. This ambiguity is very similar to the ambiguity (scheme-dependence) one encounters in defining composite operators in ordinary perturbative quantum field theory. In the U(1) theory, the operator S can mix with the operator $q\mathbb{I}$ (that we note simply by q) because their $\mathrm{U}(1)_{\mathrm{R}}$ charges (2) turn out to be the same (equal to three) when $N = 1$. Eq. (2.65) simply means that we have chosen a scheme in which in the quantum theory the operator S is *defined* to be $q\mathbb{I}$. The condensate (2.65) is thus completely fake, it comes from a mixing with the identity operator!

We hope that the above example, though essentially trivial, already shows the interest in working with the chiral ring. The main lesson is that the commonly used tools, like the effective superpotentials, can contain a lot of redundant and completely unphysical information that obscure the physics, which is unlike the chiral ring A. The U(1) theory is of course extreme; in this case the superpotential (2.64) is totally arbitrary and entirely without physical content.

Example 9. Let us now look at the U(N) gauge theory with one adjoint ϕ already discussed in section 2.2.2, example 3. We can decompose the chiral ring of this model according to the grading associated to the $\mathrm{U}(1)'_{\mathrm{R}}$ symmetry (2.31),

$$\mathsf{A} = \bigoplus_{n\in\mathbb{N}} \mathsf{A}_n \,. \tag{2.69}$$

Let us discuss the subring A_0 [21]. It is generated by the operators

$$u_k = \mathrm{Tr}\,\phi^k \,. \tag{2.70}$$

Because ϕ is a $N \times N$ matrix, the u_k are not all independent. There exists polynomial constraints of the form

$$u_{N+p} = Q_p(u_1, \ldots, u_N)\,, \quad p \geq 1 \tag{2.71}$$

that show that only $u_1, \ldots, u_N$ are independent.

In the literature, it is often claimed that the relations (2.71), which are trivial classical identities, "are corrected by instantons." The quantum relations would then take a corrected form,

$$u_{N+p} = \tilde{Q}_p(u_1, \ldots, u_N; q)\,, \quad p \geq 1\,, \tag{2.72}$$

where now the polynomials $\tilde{Q}_p$ depend non-trivially on q and coincide with the Q_p when $q = 0$.

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The question we would like to answer is: are the "quantum corrections" that appear in (2.72) genuine, unambiguous physical quantum corrections? From our previous discussions, it should be clear that the answer is *no*. The chiral ring A_0 *does not depend on the form of the relations* (2.72). In all cases, A_0 is isomorphic to a simple polynomial ring

$$\mathsf{A}_0 = \mathbb{C}[q, X_1, \ldots, X_N]\,, \tag{2.73}$$

where the X_i are as usual algebraically independent variables (identified here with the u_i). The relations (2.72) are mere *definitions* of what we mean by u_k for $k > N$ in the quantum theory. These definitions can of course be totally arbitrary. They are only restricted by the U(1) symmetries of the theory (in the case at hand, the $\mathrm{U}(1)_{\mathrm{R}}$ symmetry (2) implies that $\tilde{Q}_p = Q_p$ for $p < 2N$). Clearly, and contrary to the standard claims, the relations (2.72), being arbitrary, cannot be computed in any well-defined sense in the quantum gauge theory.

Since this is at the origin of considerable confusion, let us give more concrete details. Imagine that you want to compute the expectation value $\langle u_k \rangle$, or any chiral correlator containing the operator u_k, in the quantum gauge theory, using a microscopic first principle approach as in [1−3]. A crucial part of the calculation involves integrating over the moduli space of instantons. The instanton moduli space has singularities corresponding to instantons with vanishing size. When $k < 2N$, these singularities are integrable, i.e. the integral over the moduli space with the insertion of the operator u_k is well-defined. However, when $k \geq 2N$, the singularities are no longer integrable. Typically one finds a result of the form $\infty \times 0$, the ∞ coming from the integration over the instanton size and the 0 coming from a Grassmann integral. This phenomenon is described in details in a special case for example in section VII.2 of [23].

From our previous discussion, it should not be surprising that the correlators involving u_k for $k \geq 2N$ are ill-defined. The ambiguity we find is simply the ambiguity associated with a choice of the polynomials $\tilde{Q}_p$ in (2.72). In instanton calculus, one usually proceeds by regularizing the instanton moduli space. There is an infinite number of possible inequivalent regularizations. Once regularized, the moduli space integrals are all well-defined and we find a definite answer for the correlators. To each regularization is associated a particular definition of the operators u_k for $k > N$, i.e. a particular choice for the polynomials $\tilde{Q}_p$.[8]

In essence, the above phenomenon is the same as the one encountered in perturbation theory when one defines composite operators. The definition depends on the scheme. In our case, we are dealing with chiral operators which are unambiguous at the perturbative level, but a regularization is needed at the non-perturbative level.

Of course, the physics of the gauge theory is independent of the particular regularization of the instanton moduli space that one uses. This translates in the fact that the ring (2.73) is independent of the precise form of the polynomials $\tilde{Q}_p$.

Usually, one uses the non-commutative deformation to regularize the instanton moduli

[8] So there is an injective map between the space of polynomials $\tilde{Q}_p$ and the space of regularizations of the instanton moduli space. We do not know if this map is an isomorphism.

space. In this case the generating function

$$F(z) = z^N \exp\biggl(-\sum_{k\geq 1}\frac{u_k}{kz^k}\biggr) \tag{2.74}$$

satisfies the constraint

$$F(z) + \frac{q}{F(z)} = P(z)\,, \tag{2.75}$$

for a certain degree N monic polynomial $P(z)$. The condition (2.75) is *equivalent* to a particular choice for the relations (2.72) and the polynomials $\tilde{Q}_q$ can be computed recursively by expanding the left-hand side of (2.75) at large z and using the fact that the terms with negative powers of z must vanish. Equation (2.75) can be easily solved and yields

$$F(z) = \frac{1}{2}\Bigl(P(z) + \sqrt{P(z)^2 - 4q}\Bigr)\,, \tag{2.76}$$

$$R(z) = \frac{F'(z)}{F(z)} = \sum_{k\geq 0}\frac{u_k}{z^{k+1}} = \frac{P'(z)}{\sqrt{P(z)^2-4q}}\,. \tag{2.77}$$

These formulas for the generating functions are of course well-known. They imply that $R(z)$ and $F(z)$ are well-defined meromorphic functions on the Seiberg-Witten curve

$$y^2 = P(z)^2 - 4q\,. \tag{2.78}$$

What is usually not appreciated is that this result is a consequence of an arbitrary choice for the relations (2.72) and does not contain any non-trivial physical information. Other choices for the relations are possible. For example, it is perfectly sensible to make the choice $\tilde{Q}_p = Q_p$, in which case one finds that $F(z)$ is simply a polynomial and $R(z)$ a rational function with simple poles,

$$F(z) = P(z) = \prod_{i=1}^{N}(z - z_i) \tag{2.79}$$

$$R(z) = \frac{P'(z)}{P(z)} = \sum_{i=1}^{N}\frac{1}{z - z_i}\,. \tag{2.80}$$

The interest in making the choices that lead to (2.76) and (2.77) is that the solution of the model can then be presented in an elegant way. This will be made clear in section 5.

Example 10. As a last simple example of the use of the chiral ring, let us analyse in more details the “ambiguities” pointed out in [22]. The puzzle can be presented in the following way. In supersymmetric gauge theories, there exists operators that vanish at the perturbative level but do not at the quantum level. For example, in the pure SU(N) gauge theory, $S^k = 0$ in perturbation theory as soon as $k \geq N$ whereas $S^k \neq 0$ in the full quantum theory. Let $\mathcal{O}$ be such an operator. Imagine that we add $\mathcal{O}$ to the tree-level superpotential (2.5),

$$W_{\text{tree}} \longrightarrow \tilde{W}_{\text{tree}} = W_{\text{tree}} + g\mathcal{O}\,. \tag{2.81}$$

Clearly, at the classical level, the theories described by W_{tree} and $\tilde{W}_{\text{tree}}$ are the same. However, they look different at the quantum level. It might seem that we have an amgiguity in quantizing the classical theory and that new types of theories, corresponding to different "UV completions" in the language of [22], can be defined.

How do we solve the puzzle using the notion of the chiral ring? Any operator $\mathcal{O}$ can be written in the form (2.41). The fact that the operator vanishes simply means that the polynomial $\rho_{\mathcal{O}}$ is proportional to the instanton factors q_α. Thus adding the term $g\mathcal{O}$ to W_{tree} is *equivalent* to adding a term $g\rho_{\mathcal{O}}(\mathcal{O}_1,\ldots,O_n)$, which simply amounts to a $\boldsymbol{q}$-dependent redefinition of some of the couplings g_k appearing in the standard classical tree level superpotential (2.5). So the theory with $\tilde{W}_{\text{tree}}$ is not a new theory. It is simply a standard theory written in terms of an unusual parametrization, for which the tree-level couplings depend artificially on the instanton factors.

For example, in the pure $\mathrm{SU}(N)$ gauge theory, the most general classical tree-level superpotential that can be considered is

$$W_{\text{tree}} = \sum_{k=1}^{N-1} g_k S^k\,. \tag{2.82}$$

Taking into account (2.54), we see that adding a term of the form gS^{rN+s} with $0\leq s<N$ in the quantum theory is simply equivalent to redefining $g_k\rightarrow g_k+q^r g\delta_{ks}$ in (2.82).

3. The chiral ring and phases

We now have all the necessary tools to study the phases of the super Yang-Mills theories. An interesting feature that was pointed out in [5] is that in a given phase, there are new relations between chiral operators that come on top of the operator relations that we have discussed in section 2.5. The authors of [5] proposed that these phase-dependent relations may be used to distinguish the phases. One of our goal in the following is to make this idea precise. We shall see that indeed, individual phases are *characterized* by a set of phase-dependent relations. Quite remarkably, there are priviliged operators in each phase, that we call primitive operators, such that the full set of relations in a phase can be reduced to a *single* polynomial equation satisfied by any of the primitive operator.

We start in 3.1 by giving a physically-motivated definition of what is meant by "being in the same phase." We then proceed in 3.2 and 3.3 to study the mathematical consequences, making a direct link between the decomposition of the polynomial equations of theorem 2 into irreducible components and the existence of distinct phases. Eventually, we are led to a very simple description of the individual phases in terms of primitive operators which is explained in 3.4. All these results have a very natural geometric interpretation discussed in 3.5.

For the study of the phases it is simpler mathematically to use the chiral ring $\mathbb{A}$ defined in (2.49) instead of A and thus we shall do so in the following unless explicitly stated otherwise.

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3.1 Phases and analytic continuations

Phases are characterized by the fact that they cannot change under a smooth deformation. In other words, if we start with some given parameters $\boldsymbol{g}$ and $\boldsymbol{q}$ and in a given vacuum $|i\rangle$, then by smoothly varying the parameters we must remain in the same phase. By allowing the most general analytic continuations, we can then explore the full phase diagram of the theory. As explained in section 2.3.2, an analytic continuation can induce a permutation of the vacua.

Definition 4. The *monodromy group* of the theory is the group generated by the permutations of the vacua obtained by performing analytic continuations along arbitrary closed loop in the theory space parametrized by $(\boldsymbol{g},\boldsymbol{q})$.

Definition 5. A *phase* of a supersymmetric gauge theory is defined to be an orbit of the monodromy group acting on the set of vacua.

We thus have the following decomposition,

$$\big\{|i\rangle\,,\ 1\leq i\leq v\big\} = \bigcup_{p=1}^{\Phi} |p)\,, \tag{3.1}$$

where we have denoted by $|p)$ the orbits.

The definition 5 is *analytic* in nature. As already emphasized in section 1, using a direct analytic approach to compute the phase structure of the theory is in general extremely difficult because the analytic structure of the expectation values $\langle\mathcal{O}\rangle(\boldsymbol{g},\boldsymbol{q})$, for which explicit formulas are usually not known, can be very complicated. Our goal in the following is to develop an *algebraic* point of view which turns out to be very powerful.

3.2 Irreducible polynomials and phases

3.2.1 The fundamental example

Let us assume for the moment that the chiral ring is generated by a single operator $\mathcal{O}$. This might seem to be a gross oversimplification, but it will become clear in 3.4 that this is not so and that most of the relevant features can be described by making this assumption. The chiral ring is thus of the form

$$\mathbb{A} = \mathsf{k}[\mathcal{O}] = \mathsf{k}[X]/\mathscr{I}\,. \tag{3.2}$$

The ideal $\mathscr{I}$ is always generated by a single polynomial in this case (one says that the ring $\mathsf{k}[X]$ is principal) which is obviously the degree v polynomial $P_{\mathcal{O}}$ of theorem 2,

$$\mathscr{I} = (P_{\mathcal{O}})\,. \tag{3.3}$$

The expectation values $\langle i|\mathcal{O}|i\rangle$ in the v vacua of the theory correspond to the v roots of the equation

$$P_{\mathcal{O}}(z) = \sum_{k=0}^{v} a_k(\boldsymbol{g},\boldsymbol{q})z^{v-k} = 0\,. \tag{3.4}$$

According to definition 5, the phase structure of the theory can be computed by finding how the roots of the polynomial (3.4) are permuted when the parameters $\boldsymbol{g}$ and $\boldsymbol{q}$ are varied arbitrarily. However, instead of focusing on these *analytic* properties, it turns out to be much more fruitful to study the *arithmetic* properties of the polynomial $P_{\mathcal{O}}$.

Let us start with a basic definition. A polynomial $P \in \mathsf{k}[X]$ is said to be *irreducible* if it cannot be written as the product of two other non-trivial polynomials in $\mathsf{k}[X]$. In other words, if $P = RS$ with $R, S \in \mathsf{k}[X]$ then either R or S must be in k. Let us note that the property of irreducibility strongly depends on the base field k.

Any polynomial in $\mathsf{k}[X]$ has a prime decomposition. In particular, the polynomial $P_{\mathcal{O}}$ can be decomposed in a unique way (up to trivial multiplications by non-zero elements of k) as the product of relatively prime irreducible polynomials $P_p \in \mathsf{k}[X]$ of degree $v_p \geq 1$,

$$P_{\mathcal{O}} = \prod_{p=1}^{\Phi} P_p^{n_p} \,. \tag{3.5}$$

The integers n_p must be equal to one, since otherwise $\prod_{p=1}^{\Phi} P_p$ would be a nilpotent element of $\mathbb{A}$, in contradiction with the discussion of section 2.5.4. This decomposition in irreducible parts is of fundamental interest to us because of the following theorem.

Theorem 6. *Each phase of the theory is associated with an irreducible factor P_p in the prime decomposition of the polynomial $P_{\mathcal{O}}$ over the field* k. *In particular, in* (3.1) *the phase* $|p)$ *contains the vacua associated with the roots of the polynomial P_p.*

This result shows that one can use algebraic techniques to study the phases of the gauge theories. This is extremely useful because in many cases it is much easier to prove that a polynomial is irreducible, or to find the decomposition into irreducible factors, than to study the analytic properties of the roots.

Let us prove theorem 6. To solve the equation (3.4), we can solve the Φ algebraic equations

$$P_p(z) = \sum_{k=0}^{v_p} a_{p,k}(\boldsymbol{g}, \boldsymbol{q}) z^{v_p - k} = 0 \tag{3.6}$$

independently. Let us decompose the set of vacua as

$$\bigl\{ |i\rangle \,,\ 1 \leq i \leq v \bigr\} = \bigcup_{p=1}^{\Phi} \bigl\{ |p, i\rangle \,,\ 1 \leq i \leq v_p \bigr\} \,, \tag{3.7}$$

in such a way that the expectations values $\langle p, i|\mathcal{O}|p, i\rangle = \mathcal{O}_{p,i}(\boldsymbol{g}, \boldsymbol{q})$ be the roots of P_p. Let us now perform an analytic continuation of $\mathcal{O}_{p,i}(\boldsymbol{g}, \boldsymbol{q})$ along an arbitrary closed loop in the $(\boldsymbol{g}, \boldsymbol{q})$-space. *Because the coefficients $a_{p,k}$ in* (3.6) *are in* k, they are single-valued functions of the parameters. Thus after the analytic continuation the polynomial P_p remains the same. This implies that the analytic continuation of $\mathcal{O}_{p,i}$ must still be a root of P_p: the monodromy group acts by permuting the roots of the individual irreducible factors P_p, but *cannot mix the roots of different factors.* In other words, vacua $|p, i\rangle$ and $|p', i'\rangle$ for $p \neq p'$ must be in different phases.

Conversely let us show that all the vacua $|p,i\rangle$, for $1 \le i \le v_p$, are in the same phase. If this were not the case, then the monodromy group would have distinct orbits when acting on the roots of P_p. To each orbit, one can associate a polynomial whose roots correspond to the vacua in the orbit. Using an argument along the lines of section 2.4.2, one can show that these polynomials are in $\mathsf{k}[X]$. They would thus provide a non-trivial decomposition of P_p over k, which is impossible.

All the explicit examples we shall be dealing with in the present paper correspond to $\mathsf{k} = \mathbb{C}(\boldsymbol{g}, \boldsymbol{q})$. One useful elementary tool to study irreducibility properties of polynomial over this field is to use the following lemma.

Lemma 7. *Let* $\mathsf{a} = \mathbb{C}[\boldsymbol{g}, \boldsymbol{q}]$ *and* $\mathsf{k} = \mathbb{C}(\boldsymbol{g}, \boldsymbol{q})$. *Then if* $P \in \mathsf{a}[X]$, $\deg P \ge 1$, *is irreducible over* a *it is also irreducible over* k.

The proof can be found in standard textbooks. The result is used as follows. Imagine that you want to prove the irreducibility of $P \in \mathsf{k}[X]$ over k. We can always factorize $P(X) = a\tilde{P}(X)$ with $a \in \mathsf{k}$ and $\tilde{P} \in \mathsf{a}[X]$, where the coefficients of $\tilde{P}$ are relatively prime in a. Clearly the irreducibility of P over k is equivalent to the irreducibility of $\tilde{P}$ over k. We thus have to study the possible factorizations $\tilde{P} = QR$ over $\mathsf{k}[X]$. The lemma shows that the coefficients of Q and R can be restricted to be in a instead of k without loss of generality.

Example 11. Let us study the phase structure of the pure $\mathrm{SU}(N)$ gauge theory, for which the chiral ring is generated by a single field S. From (2.37) we know that the relevant polynomial is $P_S(X) = X^N - q$. We have to study the possible factorizations $P_S = QR$ in $\mathbb{C}(q)[X]$. From the lemma 7, we can assume that Q and R are in $\mathbb{C}[q][X]$. Since P_S, viewed as a polynomial in q, is of degree one, either R or Q, say R, does not depend on q. By setting $q = 0$ in the factorization condition, one sees that R is necessarily proportional to X^r for some $r \ge 0$ and that actually $r = 0$ since zero is not a root of P_S. Thus $R \in \mathbb{C}$ which proves that P_S is irreducible. Thus the pure gauge theory has only one phase.

Of course the result in this case follows trivially from the analytic method, because the equation $P_S = 0$ can be solved explicitly, see (2.59). All the vacua $|k\rangle$ can be smoothly connected to each other by analytic continuation: $|k\rangle \to |k+s\rangle$ by encircling s times the origin in the q-plane, $q \to e^{2i\pi s} q$. The algebraic approach is useful when explicit formulas for the roots do not exist (or are too complicated), see sections 4 and 5.

3.2.2 Operator relations in a phase

The description of the phases given by theorem 6 in terms of the decomposition $P_{\mathcal{O}} = \prod_p P_p$ has an interesting consequence. The expectation values of $\mathcal{O}$ satisfy

$$P_{\mathcal{O}}(\langle \mathcal{O} \rangle) = 0 \tag{3.8}$$

in all the vacua of the theory and we thus have an operator relation $P_{\mathcal{O}}(\mathcal{O}) = 0$ in the sense of definition 1 in section 2.5.2. On the other hand, in the phase $|p)$, the expectation values $(p|\mathcal{O}|p)$ (by which we mean the expectation values in any of the vacua belonging to the phase $|p)$) satisfy the *stronger* constraint

$$P_p\big((p|\mathcal{O}|p)\big) = 0\,. \tag{3.9}$$

This naturally leads to the following definitions.

Definition 6. Let $\mathcal{O}^{(1)}, \ldots, \mathcal{O}^{(p)}$ be chiral operators. An *operator relation in a phase* $|\varphi)$ is a polynomial equation of the form

$$P(\mathcal{O}^{(1)}, \ldots, \mathcal{O}^{(p)}) = 0\,, \quad P \in \mathsf{a}[X_1, \ldots, X_p]\,, \tag{3.10}$$

such that $P(\langle i|\mathcal{O}^{(1)}|i\rangle, \ldots, \langle i|\mathcal{O}^{(p)}|i\rangle)$ identically vanishes in all the vacua $|i\rangle$ belonging to the phase $|\varphi)$.

Definition 7. Let $\mathscr{I}_{|\varphi)}$ be the ideal generated by all the operator relations in the phase $|\varphi)$. Clearly, $\mathscr{I} \subset \mathscr{I}_{|\varphi)}$ and thus $\mathscr{I}_{|\varphi)}$ can be seen as an ideal of the chiral ring A defined in (2.45) (or of $\mathbb{A}$ defined in (2.49)). The quantum chiral rings in the phase $|\varphi)$ are then defined by

$$\mathsf{A}_{|\varphi)} = \mathsf{A}/\mathscr{I}_{|\varphi)}\,, \quad \mathbb{A}_{|\varphi)} = \mathbb{A}/\mathscr{I}_{|\varphi)}\,. \tag{3.11}$$

The rings $\mathsf{A}_{|\varphi)}$ and $\mathbb{A}_{|\varphi)}$ have remarkable properties that are discussed in the following sections.

3.2.3 On the use of irreducible polynomials

Consider now a general supersymmetric gauge theory. Imagine that we want to demonstrate that two vacua $|i\rangle$ and $|j\rangle$ belong to the same phase. For example, in theories with fundamentals, we would like to show that the "confining" and the "Higgs" vacua are in the same phase. The discussion in the previous subsections suggests the following strategy: find an operator $\mathcal{O}$ such that the vacua $|i\rangle$ and $|j\rangle$ are associated with two roots of the *same* irreducible factor in the decomposition of $P_{\mathcal{O}}$. This approach turns out to be a very efficient way to make the proof.

Let us be more precise in the case of the theory (2.20). As we have already explained, this model is the natural arena to study the possible transitions from the Higgs to the confining regime. The claim is that all the vacua of rank one of the model should be in the same phase, irrespective of the pattern of gauge symmetry breaking. This is a direct consequence of the following result.

Lemma 8. *When $N_{\mathrm{f}} < N$, the polynomial P_S for the glueball superfield S in the model* (2.20) *is irreducible over $\mathbb{C}[\mu, m_1, \ldots, m_{N_{\mathrm{f}}}, q]$. When $N_{\mathrm{f}} \geq N$, $P_S(X) = X^{\binom{N_{\mathrm{f}}}{N}}\tilde{P}_S(X)$, where $\tilde{P}_S$ is irreducible over $\mathbb{C}[\mu, m_1, \ldots, m_{N_{\mathrm{f}}}, q]$. The factor $X^{\binom{N_{\mathrm{f}}}{N}}$ corresponds to a purely classical part associated with the vacua of rank zero and the other factor $\tilde{P}_S$ to the Higgs/confining vacua of rank one.*

A reader that would be interested specifically in the problem of the equivalence between the Higgs and confining regimes may now jump to section 4 where an explicit construction of the polynomial P_S and the proof of lemma 8 can be found.

3.3 The prime decomposition

In section 3.2.1 we assumed that the chiral ring were generated by a single operator $\mathcal{O}$. The phase structure of the model is then given by the decomposition of the polynomial $P_{\mathcal{O}}$ into irreducible factors.

How can we generalize this result to the generic case with a finite number of generators $\mathcal{O}_1, \ldots, \mathcal{O}_n$?

3.3.1 Phases and operator relations

First, we have the analogue of theorem 2 for a given phase.

Proposition 9. *Let $|\varphi)$ be a phase that contains v_φ vacua. Then any chiral operator $\mathcal{O}$ satisfies a degree v_φ operator relation in the phase $|\varphi)$ of the form $P_{\mathcal{O}}^{|\varphi)}(\mathcal{O}) = 0$, $P_{\mathcal{O}}^{|\varphi)} \in \mathsf{a}[X]$.*

The proof is strictly similar to the proof of theorem 2 and we let the details to the reader. From proposition 9 one can directly derive the analogue of proposition 4 in section 2.5.

Proposition 10. *The full solution of the theory in the chiral sector in the phase $|\varphi)$ is coded in the chiral ring $\mathsf{A}_{|\varphi)}$ (or $\mathbb{A}_{|\varphi)}$) in the phase $|\varphi)$, i.e. we can compute the expectation values $\langle\mathcal{O}\rangle$ in any vacuum belonging to the phase $|\varphi)$ and for any chiral operator $\mathcal{O}$ from the knowledge of the ring $\mathsf{A}_{|\varphi)}$ (or $\mathbb{A}_{|\varphi)}$).*

This result makes very precise the idea proposed in [5]. If $\mathscr{I}_{|\varphi)} = \mathscr{I}$ then clearly the theory has only one phase. However, in general one has a strict inclusion $\mathscr{I} \subsetneq \mathscr{I}_{|\varphi)}$ and there are new operator relations valid only in the phase $|\varphi)$. Moreover these new relations completely determine the expectation values in the phase.

One may ask if the chiral ring $\mathbb{A}_{|\varphi)}$ (or equivalently the operator relations in the phase $|\varphi)$) could be considered to be like an "order parameter" characterizing the phase in some fundamental way. The answer to this question is no. This is best illustrated on an example, so let us consider the $\mathrm{U}(N)$ theory with one adjoint ϕ and tree-level superpotential (2.10). If p is the degree of W', as in (2.11), then the theory has $v_\varphi = pN$ vacua of rank one (see (2.15) and (2.19)) corresponding to an unbroken gauge group. It is not difficult to show that all these vacua are in the same phase $|\varphi)$ (see section 5). This phase does not depend on the value of p: increasing p amounts to turning on some couplings and the new vacua that then appear can be smoothly connected to the old vacua. Physically, this phase simply corresponds to the standard confining phase of the pure super Yang-Mills theory. On the other hand, the structure of $\mathbb{A}_{|\varphi)}$ does depend on p. This can be seen, for example, from the fact that the dimension of $\mathbb{A}_{|\varphi)}$ viewed as a k vector space is equal to the number v_φ of vacua in the phase $|\varphi)$ (this is a general result that will be derived in section 3.4) and that this number depends on p.

The lesson is that it is not trivial to obtain new kinds of order parameters that can help in distinguishing the phases at a fundamental level. In particular, the chiral ring itself is not a good candidate, because physically equivalent phases can have distinct chiral rings. Nevertheless, our formalism can be used to shed an interesting new light on this question, using Galois theory. This is explained in details in a separate publication [6].

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3.3.2 The chiral field

The chiral ring in a phase has a crucial property that plays in particular a prominent rôle in [6].

Proposition 11. *Let $|\varphi)$ be a phase. The ideal $\mathscr{I}_{|\varphi)} \subset \mathsf{A}$ is prime. Equivalently, the ring $\mathsf{A}_{|\varphi)}$ is an integral domain.*

When $\mathscr{I}_{|\varphi)}$ is generated by a single polynomial P (as in the case studied in section 3.2.1), the condition that $\mathscr{I}_{|\varphi)}$ is prime is equivalent to the condition that the polynomial P is irreducible. In general, it means that if $RS \in \mathscr{I}_{|\varphi)}$, then either $R \in \mathscr{I}_{|\varphi)}$ or $S \in \mathscr{I}_{|\varphi)}$. Clearly this is equivalent to the fact that $A_{|\varphi)}$, which is isomorphic to $\mathsf{a}[X_1, \ldots, X_n]/\mathscr{I}_{|\varphi)}$, is an integral domain: in $\mathsf{A}_{|\varphi)}$, $\mathcal{AB} = 0$ implies that either $\mathcal{A} = 0$ or $\mathcal{B} = 0$.

It is not difficult to understand why $A_{|\varphi)}$ must be an integral domain. Pick two operators $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{AB} = 0$. This is equivalent to the fact that the expectation value of $\mathcal{AB}$ in any vacuum belonging to the phase $|\varphi)$ vanishes,

$$(\varphi|\mathcal{AB}|\varphi) = 0 = (\varphi|\mathcal{A}|\varphi)(\varphi|\mathcal{B}|\varphi)\,. \tag{3.12}$$

If $\mathcal{A}$ and $\mathcal{B}$ are both zero in $\mathsf{A}_{|\varphi)}$ then there is nothing to prove. Let us thus assume that $\mathcal{A} \neq 0$ and let us prove that this implies that $\mathcal{B} = 0$. The condition $\mathcal{A} \neq 0$ in $\mathsf{A}_{|\varphi)}$ means that there exists at least one vacuum $|i\rangle$ in the phase $|\varphi)$ such that $\langle i|\mathcal{A}|i\rangle \neq 0$. Equation (3.12) then automatically implies that $\langle i|\mathcal{B}|i\rangle = 0$. Let now $|j\rangle$ be an arbitrary vacuum in $|\varphi)$. Because $|i\rangle$ and $|j\rangle$ are in the same phase, the expectation value $\langle j|\mathcal{B}|j\rangle$ can be obtained by analytic continuation from $\langle i|\mathcal{B}|i\rangle = 0$ and is thus automatically zero. The conclusion is that the expectation value of $\mathcal{B}$ vanishes in all the vacua of the phase $|\varphi)$, i.e. that $\mathcal{B} = 0$ in $\mathsf{A}_{|\varphi)}$.

It is important to realize that this property is very special to the chiral rings *in a given phase* and that it is not shared by the chiral ring A (or $\mathbb{A}$) in general. For example, in the case studied in 3.2.1, $P_{\mathcal{O}} = \prod_p P_p = 0$ in A, but the individual irreducible factors P_p are all non-zero in A if there is more than one phase. It is actually not difficult to show that in general A is an integral domain if and only if the theory is realized in a single phase.

An even stronger property is true for the ring $\mathbb{A}_{|\varphi)}$.

Theorem 12. *Let $|\varphi)$ be a phase. The ideal $\mathscr{I}_{|\varphi)} \subset \mathbb{A}$ is maximal. Equivalently, the ring $\mathbb{A}_{|\varphi)}$ is a field, which is the field of fractions of $\mathsf{A}_{|\varphi)}$.*

Being a field is a very remarkable property for an algebra of operator. It means that every non-zero operator has an inverse. Very concretely, if $\mathcal{O} = \rho_{\mathcal{O}}(\mathcal{O}_1, \ldots, \mathcal{O}_n)$, $\rho_{\mathcal{O}} \in \mathsf{k}[X_1, \ldots, X_n]$, is an arbitrary non-zero operator, then it is always possible to find another non-zero operator $\mathcal{O}' = \rho_{\mathcal{O}'}(\mathcal{O}_1, \ldots, \mathcal{O}_n)$, $\rho_{\mathcal{O}'} \in \mathsf{k}[X_1, \ldots, X_n]$, such that $\mathcal{OO}' = \rho_{\mathcal{O}}(\mathcal{O}_1, \ldots, \mathcal{O}_n)\rho_{\mathcal{O}'}(\mathcal{O}_1, \ldots, \mathcal{O}_n) = 1$ in $\mathbb{A}_{|\varphi)}$. In other words, thank's to the additional operator relations that are satisfied in a given phase, an arbitrary rational function in the generators $\mathcal{O}_1, \ldots, \mathcal{O}_n$ can always be shown to be equal to a particular polynomial.

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Example 12. Before we proceed to the proof, let us illustrate this result for the pure $\mathrm{SU}(N)$ gauge theory. This theory is realized in a single phase and thus theorem 12 implies that the chiral ring

$$\mathbb{A} = \mathbb{C}(q)[S]/(S^N - q) \tag{3.13}$$

itself should be a field. Indeed, using the operator relation $S^N = q$, it is clear that the inverse of the glueball operator is simply given by $S^{-1} = S^{N-1}/q$. The inverse of an arbitrary operator of the form $\rho(S)$ for $\rho \in \mathbb{C}(q)[X]$ can also be constructed straightforwardly using the euclidean division algorithm.

The simplest proof of theorem 12 relies on the fact that $\mathbb{A}_{|\varphi)}$ is a finite dimensional k vector space. Indeed $\mathbb{A}$ itself is finite dimensional, as explained in section 2.5.4. Assume then that $\mathcal{O} \in \mathbb{A}_{|\varphi)}$ is non zero and consider the k-linear map $\mathcal{O}' \mapsto \mathcal{O}\mathcal{O}'$. This map is injective because $\mathbb{A}_{|\varphi)}$ is an integral domain (using exactly the same argument that shows that $\mathsf{A}_{|\varphi)}$ is an integral domain). Being a linear map of a finite dimensional vector space, it must also be surjective and thus in particular its image contains the identity. This implies that $\mathcal{O}$ has an inverse as was to be shown.

The ring $\mathsf{A}_{|\varphi)}$, being an integral domain, has a field of fractions $\mathrm{Frac}(\mathsf{A}_{|\varphi)})$ which is the smallest field containing $\mathsf{A}_{|\varphi)}$ and which is built by considering fractions of the elements of $\mathsf{A}_{|\varphi)}$. Clearly $\mathbb{A}_{|\varphi)} \subset \mathrm{Frac}(\mathsf{A}_{|\varphi)})$ and since $\mathbb{A}_{|\varphi)}$ is a field the inclusion must be an equality.

We shall have more to say about the chiral field $\mathbb{A}_{|\varphi)}$ in section 3.4.

3.3.3 The prime decomposition

We have seen that phases are characterized by prime ideals $\mathscr{I}_{|\varphi)} \supset \mathscr{I}$ describing the operator relations in the given phase. When there is only one generator $\mathscr{I} = (P_{\mathcal{O}})$ as in (3.3), these prime ideals are generated by the irreducible factors of $P_{\mathcal{O}}$. In the general case, the decomposition of a given polynomial into irreducible factors is replaced by the decomposition of a given radical ideal into prime ideals.

Theorem 13. *Let $\mathscr{I}$ be the ideal of operator relations. Then one can write in a unique way*

$$\mathscr{I} = \bigcap_{p=1}^{\Phi} \mathscr{I}_{|p)} \tag{3.14}$$

where the $\mathscr{I}_{|p)}$ are prime ideals such that $\mathscr{I}_{|p)} \not\subset \mathscr{I}_{|p')}$ if $|p) \neq |p')$. The decomposition (3.14) *corresponds to the decomposition* (3.1) *into phases, the ideal $\mathscr{I}_{|p)}$ being generated by the operator relations in the phase $|p)$.*

This theorem provides a completely general algebraic method to obtain the phase structure of a given model. Of course, computing the prime decomposition of a radical ideal $\mathscr{I}$ is non-trivial. It is one of the basic problem of computational commutative algebra. A very useful fact is that sophisticated algorithms that perform this decomposition have been implemented on computer algebra systems like SINGULAR [13] that are heavily used in section 5 to study the phases of the model (2.10).

The theorem 13 is standard and a proof can be found in the textbooks [17]. However, since this is a fundamental result for us, and also because the textbooks usually deal

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with the most general case of the primary decomposition of an arbitrary ideal instead of the simpler prime decomposition of a radical ideal that we need, let us briefly sketch the argument. If $\mathscr{I}$ is prime then the theory has only one phase and there is nothing to do. If $\mathscr{I}$ is not prime, then we can find an operator relation of the form $ab \in \mathscr{I}$ but with $a \notin \mathscr{I}$ for example. It is then natural to impose new operator relations corresponding to $a = 0$ or $b = 0$, which are associated with the radical ideals $\mathscr{I}_1 = r(\mathscr{I} + (a))$ and $\mathscr{I}_2 = r(\mathscr{I} + (b))$. Using the fact that $\mathscr{I}$ is radical, it is not difficult to check that $\mathscr{I} = \mathscr{I}_1 \cap \mathscr{I}_2$. If $\mathscr{I}_1$ and $\mathscr{I}_2$ are prime, then we have finished. Otherwise, we can repeat the above argument and further decompose the ideals $\mathscr{I}_1$ and/or $\mathscr{I}_2$. Eventually, this process must terminate because A is noetherian and we find the decomposition (3.14). If we had two decompositions for $\mathscr{I}$ based on prime ideals $(\mathscr{I}_{|p)})$ and $(\mathscr{J}_{|q)})$, then it is easy to see that $\cap_p \mathscr{I}_{|p)} = \cap_q \mathscr{J}_{|q)}$ implies that, for any p, $\mathscr{I}_{|p)} = \cap_q (\mathscr{I}_{|p)} + \mathscr{J}_{|q)})$. Because $\mathscr{I}_{|p)}$ is prime this implies that there exists q for which $\mathscr{I}_{|p)} = \mathscr{I}_{|p)} + \mathscr{J}_{|q)}$, i.e. $\mathscr{J}_{|q)} \subset \mathscr{I}_{|p)}$. Similarly $\mathscr{I}_{|p')} \subset \mathscr{J}_{|q)}$ for some p'. The requirement that $\mathscr{I}_{|p)} \not\subset \mathscr{I}_{|p')}$ if $|p) \neq |p')$ then shows that $p = p'$ and $\mathscr{I}_{|p)} = \mathscr{J}_{|q)}$, proving the uniqueness of the decomposition.

3.4 Primitive operators

We are now going to complete our toolkit with a remarkable result that drastically simplifies the description of individual phases.

3.4.1 The structure of the chiral field in a phase

Theorem 14. *Let $|\varphi)$ be a phase that contains v_φ vacua. The associated chiral ring $\mathbb{A}_{|\varphi)}$ is generated by a single operator $\mathcal{O}_\varphi$, called a primitive operator for the phase $|\varphi)$. This operator satisfies an operator relation in the phase $|\varphi)$ of the form $P^{|\varphi)}_{\mathcal{O}_\varphi}(\mathcal{O}_\varphi) = 0$, where $P^{|\varphi)}_{\mathcal{O}_\varphi} \in \mathsf{k}[X]$ is irreducible and of degree v_φ. In particular,*

$$\mathbb{A}_{|\varphi)} = \mathsf{k}[X]/(P^{|\varphi)}_{\mathcal{O}_\varphi}) \tag{3.15}$$

and $\dim_{\mathsf{k}} \mathbb{A}_{|\varphi)} = v_\varphi$.

This theorem shows that the physics of a given phase is always entirely coded in the expectation value of a *single* chiral operator $\mathcal{O}_\varphi$. All we need to know is the irreducible polynomial equation satisfied by this expectation value. All the other expectation values in the phase are simple polynomials in $\langle \mathcal{O}_\varphi \rangle$ with coefficients in k.

Example 13. Let us consider a gauge theory that is realized in a single phase. Then theorem 14 implies that the chiral ring of such a theory is generated by a single operator, as in the case of the pure gauge theory. This is clearly an extremely powerful result. For example, from lemma 8 we can deduce that the glueball operator S is a primitive operator for the model (2.20) when $N_{\mathrm{f}} < N$. In particular, this implies that all the operators of the form $\mathrm{Tr}\, \phi^k$, $\mathrm{Tr}\, W^\alpha W_\alpha \phi^k$ and $\tilde{Q}^f \phi^k Q_{f'}$ for any k, are actually simple polynomials in S! We shall see this explicitly in section 4.

The theorem 14 is a direct consequence of the Primitive Element theorem whose proof (which requires some technology that we have not introduced) can be found in standard textbooks [17]. The somewhat simplified version that we need is as follows.

Lemma 15. *Let* k *be a field of characteristic zero. Let* $\mathsf{K} \supset \mathsf{k}$ *be a finitely generated and algebraic field extension (an algebraic extension is such that any element of* K *satisfies an algebraic equation with coefficients over* k*). Then there always exists an element* $\alpha \in \mathsf{K}$ *such that* $\mathsf{K} = \mathsf{k}(\alpha)$ *is the field generated by* α *over* k*.*

In our case k is the field of parameters which is always of characteristic zero since it contains $\mathbb{C}$ as a subfield. The extension field we consider is $\mathbb{A}_{|\varphi)} \supset \mathsf{k}$. It is finitely generated since the chiral ring $\mathbb{A}$ itself is and it is algebraic by proposition 9.

3.4.2 A simple test for a primitive operator

There are in general many primitive operators in a given phase. More precisely, we have the following proposition.

Proposition 16. *Let* $\mathcal{O}$ *be a chiral operator such that* $P_{\mathcal{O}}^{|\varphi)}(\mathcal{O}) = 0$ *in the phase* $|\varphi)$ *(see proposition 9). Then* $\mathcal{O}$ *is a primitive operator in the phase* $|\varphi)$ *if and only if* $P_{\mathcal{O}}^{|\varphi)}$ *is irreducible.*

This is an easy consequence of theorem 14. Indeed, if we denote by $\mathsf{k}(\mathcal{O})$ the subfield of $\mathbb{A}_{|\varphi)}$ generated by $\mathcal{O}$ over k, then $\dim_{\mathsf{k}} \mathsf{k}(\mathcal{O}) = \deg P_{\mathcal{O}}^{|\varphi)}$ because $P_{\mathcal{O}}^{|\varphi)}$ is irreducible. This shows that $\dim_{\mathsf{k}} \mathsf{k}(\mathcal{O}) = v_\varphi = \dim_{\mathsf{k}} \mathbb{A}_{|\varphi)}$ and thus that $\mathsf{k}(\mathcal{O}) = \mathbb{A}_{|\varphi)}$.

Physically, the primitive operators are the operators that "distinguish" all the vacua of the phase: by analytic continuation their expectation value can have v_φ distinct semi-classical expansions.

Proposition 17. *Let* $\mathcal{O}$ *be a chiral operator and* $|\varphi)$ *be a phase containing* v_φ *vacua. Assume that for some given values of the parameters, the* v_φ *expectations values* $\langle i|\mathcal{O}|i\rangle$ *for* $|i\rangle \in |\varphi)$ *are distinct complex numbers. Then* $\mathcal{O}$ *is a primitive operator in the phase* $|\varphi)$*.*

This result provides a simple numerical test to show that an operator is primitive in a given phase, because the expectation values $\langle i|\mathcal{O}|i\rangle$ for some given parameters $\boldsymbol{g}$ and $\boldsymbol{q}$ can be found by solving numerically the system of algebraic equations that corresponds to the prime ideal $\mathscr{I}_{|\varphi)}$ defining the phase $|\varphi)$.

3.4.3 The quantum effective superpotential

A very natural way to construct a primitive operator is as follows. The quantum effective superpotential $W_{\text{eff}}^{|i\rangle}(\boldsymbol{g}, \boldsymbol{q})$ (also often denoted as W_{low} in the literature) is defined by performing the path integral in a given vacuum $|i\rangle$ and extracting the F-terms from the resulting effective action for the background chiral superfields $\boldsymbol{g}$ and $\boldsymbol{q}$. The fundamental property of W_{eff} is that its derivatives with respect to the couplings yield the associated expectation values. For example, with a tree-level superpotential (2.5),

$$\frac{\partial W_{\text{eff}}^{|i\rangle}}{\partial g_k} = \langle i|\mathcal{O}_k|i\rangle \tag{3.16}$$

and we also have

$$\frac{\partial W_{\text{eff}}^{|i\rangle}}{\partial \ln q_\alpha} = \langle i|S_\alpha|i\rangle \tag{3.17}$$

where S_α is the glueball operator in the simple factor $\mathfrak{g}_\alpha$ of the gauge group (see section 2.1). If couplings to all the generators of the chiral ring are introduced, as we assume in this subsection, then clearly the full solution of the theory is encoded in the analytic function $W_{\text{eff}}(\boldsymbol{g},\boldsymbol{q})$.

A very nice property of the analytic function $W_{\text{eff}}(\boldsymbol{g},\boldsymbol{q})$ is that it is always given by the expectation value of a certain chiral operator,

$$W_{\text{eff}}(\boldsymbol{g},\boldsymbol{q}) = \langle \mathcal{W}\rangle\,. \tag{3.18}$$

This is a consequence of the Ward identity

$$3W_{\text{eff}} = \sum_k [g_k] g_k \frac{\partial W_{\text{eff}}}{\partial g_k} + \sum_\alpha [q_\alpha] q_\alpha \frac{\partial W_{\text{eff}}}{\partial q_\alpha}\,, \tag{3.19}$$

which follows from the $\mathrm{U}(1)_{\mathrm{R}}$ symmetry (table 2). In (3.19), we have denoted by $[g_k]$ and $[q_\alpha]$ the $\mathrm{U}(1)_{\mathrm{R}}$ charges of the various couplings. Using (3.16) and (3.17), we then obtain (3.18) with

$$\mathcal{W} = \frac{1}{3}\Bigl(\sum_k [g_k] g_k \mathcal{O}_k + \sum_\alpha [q_\alpha] S_\alpha\Bigr) \in \mathbb{A}\,. \tag{3.20}$$

As for any other chiral operator, $\mathcal{W}$ satisfies a degree v operator relation $P_{\mathcal{W}}(\mathcal{W}) = 0$. The phase structure of the theory can then always be obtained from the factorization of $P_{\mathcal{W}}$ into irreducible factors over k. Moreover, in each phase, $\mathcal{W}$ is a primitive operator. In particular, in a given phase, any chiral operator expectation value is always given in terms of the effective superpotential by a simple (phase-dependent) *polynomial* expression,

$$(\varphi|\mathcal{O}|\varphi) = T_{\mathcal{O}}^{|\varphi)}\bigl(W_{\text{eff}}^{|\varphi)}\bigr)\,,\quad T_{\mathcal{O}}^{|\varphi)} \in \mathsf{k}[X]\,. \tag{3.21}$$

Note that an expression of the form (3.21) would be valid for *any* primitive operator in each individual phases. In this sense, the notion of a primitive operator is an algebraic generalization of the notion of the quantum effective superpotential.

To finish this subsection, let us mention that the lemma 15 can be refined [17]. If a set of generators for the field K over k is known, then it can be shown that the primitive element can always be chosen to be a linear combination with coefficients in k of these generators. Eq. (3.20) shows that $\mathcal{W}$ is precisely of this form.

3.5 The geometric picture

Up to now, we have emphasized the algebraic point of view, because this is how the calculations are done in practice. However, there is a standard and elegant geometric interpretation of the results. For simplicity, let us consider the case where $\mathsf{a} = \mathbb{C}[\boldsymbol{g},\boldsymbol{q}]$. The operator relations that generate the ideal $\mathscr{I}$ can be interpreted as the defining equations for an affine algebraic variety $\mathscr{M}$. If v is as usual the number of vacua of the theory, this

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"quantum space of parameters" is a v-fold cover of the $(\boldsymbol{g}, \boldsymbol{q})$-plane. The chiral ring A defined in (2.45) is simply the ring of regular functions on $\mathscr{M}$, often called the coordinate ring of the variety $\mathscr{M}$ in the literature. The decomposition of the set of vacua into phases (3.1), or equivalently the prime decomposition of the ideal $\mathscr{I}$ (3.14), corresponds to the decomposition of the variety $\mathscr{M}$ into irreducible components,

$$\mathscr{M} = \bigcup_{p=1}^{\Phi} \mathscr{M}_{|p)} \,. \tag{3.22}$$

The ring $\mathsf{A}_{|p)}$ defined in (3.11) is the ring of regular functions on the irreducible variety $\mathscr{M}_{|p)}$. On the other hand, $\mathbb{A}_{|p)}$ corresponds to the field of rational functions on $\mathscr{M}_{|p)}$.

The existence of primitive operators (section 3.4) has also a nice geometrical interpretation. The fact that $\mathbb{A}_{|p)} = \mathsf{k}[X]/(P^{|p)})$ for a certain polynomial $P^{|p)}$ (see (3.15)) shows that the variety $\mathscr{M}_{|p)}$ can be described by the single equation $P^{|p)} = 0$. This result corresponds to a standard theorem in algebraic geometry: any irreducible affine variety is birationally equivalent to a hypersurface.

3.6 Phase transitions

The various irreducible components of $\mathscr{M}$ may intersect non-trivially. The variety $\mathscr{M}_{|p)} \cap \mathscr{M}_{|p')}$, with associated ideal of operator relation $r(\mathscr{I}_{|p)} + \mathscr{I}_{|p')})$, describes the phase transition between $|p)$ and $|p')$. Physically, these phase transitions are associated with the appearance of new massless degrees of freedom that often correspond to non-trivial IR fixed points of the gauge theory. It is actually natural to consider that the intersections between distinct phases correspond to new phases of the gauge theory. The variety $\mathscr{M}_{|p)} \cap \mathscr{M}_{|p')}$ itself can have a non-trivial decomposition in terms of irreducible components, corresponding to the prime decomposition of $r(\mathscr{I}_{|p)} + \mathscr{I}_{|p')})$. These irreducible components can themselves intersect, etc... One can also consider the intersections between more than two phases. In general, a very complex nested structure of phases and phase transitions can emerge in this way, associated with families of non-trivial superconformal fixed points. Even though this is beyond the goals of the present work, it is clear that our approach and the tools we are using are perfectly appropriate for a systematic study of this structure.

3.7 Chiral duality

At the classical level, a gauge theory is characterized by its gauge group, its matter content and its tree-level superpotential. At the quantum level, things are much more interesting. On the one hand, only gauge invariant operators make sense and thus the gauge group is no longer directly visible (the gauge group is not a physical symmetry but a redundancy in the description of the physics). On the other hand, the equations of motion derived from the tree-level superpotential are quantum corrected. The result is that two completely different looking classical theories may correspond to physically equivalent quantum theory. One then says that the theories are "dual" to each other. A weaker but very useful statement is that two theories can be physically equivalent below a certain energy scale, i.e. two distinct theories in the UV may flow to the same theory in the IR. This kind of equivalence is usually called a "Seiberg duality."

In the context of the present paper, it is very natural to study dualities between theories that have physically equivalent chiral sectors. We call a duality of this type a *chiral duality.* In general, a chiral duality is not the same as a Seiberg duality, since it also applies to cases where all the fields are massive. However, when the chiral ring is generated by massless moduli, then clearly Seiberg duality implies the equivalence of the chiral sectors of the theories. The case of massive theories can then be obtained by deformation. In practice, this can yield powerful tests of Seiberg dualities.

Each individual phase of a given theory can be considered to be a consistent quantum theory of its own and it is natural to study dualities between phases rather than between full gauge theories. We are thus led to the following definition.

Definition 8. A strong chiral duality between two phases $|p)$ and $|q)$ of two possibly distinct gauge theories is an isomorphism between the rings $\mathsf{A}_{|p)}$ and $\mathsf{A}_{|q)}$.

In the geometric language of section 3.5, the strong chiral Seiberg duality is thus an isomorphism between the affine algebraic variety $\mathscr{M}_{|p)}$ and $\mathscr{M}_{|q)}$.

As we have emphasized many times, it is very natural to allow rational combinations of the parameters to enter into the definition of the most general chiral operators. This leads to a weak form of the chiral duality.

Definition 9. A weak chiral duality (or simply a chiral duality for short) between two phases $|p)$ and $|q)$ of two possibly distinct gauge theories is an isomorphism between the fields $\mathbb{A}_{|p)}$ and $\mathbb{A}_{|q)}$.

Geometrically, the weak chiral duality between two phases $|p)$ and $|q)$ is equivalent to the birational equivalence between the associated irreducible algebraic varieties $\mathscr{M}_{|p)}$ and $\mathscr{M}_{|q)}$. This is weaker than a strong chiral duality because the invertible birational mapping $\mathscr{M}_{|p)} \to \mathscr{M}_{|q)}$ can be singular for certain values of the parameters (at the poles in the denominators). Nevertheless, the weak chiral duality ensures that the algebras of operators over k are the same in the two dual theories and thus they cannot be physically distinguished. As we illustrate below, the standard examples of Seiberg duality correspond to the weak form of definition 9.

Example 14. Let us first use a toy example to illustrate the above concepts. Let us explain how to construct chiral duals to the pure gauge theory (3.13). A dual must be in a single phase as is the original theory and thus it is described by a single primitive operator s. The ring isomorphism implies that s can be written as a polynomial in the glueball S with coefficients in $\mathbb{C}(q)$. If s then satisfies an *irreducible* polynomial equation of degree N over $\mathbb{C}(q)$, then we know that the rings of the two theories must coincide (their dimensions over $\mathbb{C}(q)$ will be the same). In this case S can also be expressed as a polynomial in s with coefficients in $\mathbb{C}(q)$, yielding the birational mapping. For example, in the case $N = 3$, consider $s = S + S^2$. The operator s satisfies the degree three equation

$$s^3 - 3qs - q - q^2 = 0 \tag{3.23}$$

as a consequence of $S^3 = q$. Using the relation between s and S, it is clear that one can interpolate smoothly between the three roots of (3.23) and thus this equation is irreducible over $\mathbb{C}(q)$. This shows immediately that the theories described by (3.23)and by $S^3 = q$ are chiral dual. The polynomial relation giving S as a function of s can be readily obtained,

$$s = S + S^2 \Longleftrightarrow S = \frac{1}{1-q}\big(2q + s - s^2\big)\,. \tag{3.24}$$

This gives the birational isomorphism between the varieties $S^3 - q = 0$ and $s^3 - 3qs - q - q^2 = 0$ (3.23).

Example 15. Let us now consider the $\mathrm{SU}(N)$ theory with N_f flavors and tree-level superpotential (2.9) (see also example 6 in section 2.4.4). We assume that $N_\mathrm{f} > 3N/2$ and we limit our discussion to the sector of zero baryonic charge for simplicity. The chiral ring is then generated by the mesonic operators $M_{f'}^{\ f}$ and by the glueball S. The operator relations read

$$m_f^{\ f''} M_{f''}^{\ f'} = N_\mathrm{f} S \delta_f^{f'}\,, \quad S^N = q \det m\,. \tag{3.25}$$

Let us also consider a different gauge theory, with gauge group $\mathrm{SU}(N_\mathrm{f} - N)$, N_f flavors of quarks q_f and $\tilde{q}^f$, one singlet $N_f^{\ f'}$ and tree-level superpotential

$$W_\mathrm{tree} = \big(q_f \tilde{q}^{f'} + m_f^{\ f'}\big) N_{f'}^{\ f}\,. \tag{3.26}$$

The chiral ring in the zero baryonic charge sector is generated by the mesons $\hat{M}_f^{\ f'} = \tilde{q}^{f'} q_f$, the singlet $N_{f'}^{\ f}$ and the glueball s. It can be argued (see for examples [24]) that the operator relations in the quantum theory read

$$\hat{M}_f^{\ f'} = -m_f^{\ f'}\,, \quad \hat{M}_f^{\ f''} N_{f''}^{\ f'} = N_\mathrm{f} s \delta_f^{f'}\,, \quad s^N = (-1)^{N_\mathrm{f}} \frac{\det m}{\hat{q}}\,, \tag{3.27}$$

where $\hat{q}$ is the instanton factor. From (3.25) and (3.27) it is clear that the two fields $\mathbb{C}(q, m_f^{\ f'})[M_f^{\ f'}, S]$ and $\mathbb{C}(\hat{q}, m_f^{\ f'})[\hat{M}_f^{\ f'}, N_{f'}^{\ f}, s]$ are isomorphic, with the identifications

$$M_f^{\ f'} = N_f^{\ f'}\,, \quad S = -s\,, \quad q = \frac{(-1)^{N_\mathrm{f} - N}}{\hat{q}}\,\cdot \tag{3.28}$$

The relations (3.28) give the birational isomorphism between the varieties defined by (3.25) and (3.27). The singularity of the mapping at $\hat{q} = 0$ corresponds to the well-known fact that the model (3.26) breaks supersymmetry at tree-level, see for example [25] for an extensive discussion.

3.8 On the semi-classical phase diagram

It is often useful to start the analysis of the phase diagram of a given gauge theory by using the weak coupling approximation. One then obtains a decomposition of the set of vacua of the theory of the form

$$\big\{|i\rangle\,,\ 1 \le i \le v\big\} = \bigcup_{p=1}^{\tilde{\Phi}} |p)_\mathrm{s.c.}\,, \tag{3.29}$$

where the "semi-classical" phases $|p)_\mathrm{s.c.}$ contain vacua that can be connected to each other in the weak coupling region. In general, the phases of the full quantum theory appearing in the decomposition (3.1) can contain several of the semi-classical phases appearing in (3.29), since vacua that cannot be smoothly related at weak coupling may be related by an analytic continuation that probe the strong coupling regime of the theory.

Let us note that explicit formulas for the chiral operator expectation values can be easily obtained at weak coupling and thus in practice the decomposition (3.29) can be most easily computed using the standard "analytic" approach. Nevertheless, it is interesting to explain how the semi-classical approximation can be interpreted in the algebraic language that we have developed so far.

It turns out that the semi-classical decomposition (3.29) corresponds to a factorization of the polynomial equations of the form

$$P_{\mathcal{O}} = \prod_{p=1}^{\tilde{\Phi}} \tilde{P}_p\,, \tag{3.30}$$

where now the factors $\tilde{P}_p$ are irreducible polynomials with coefficients in

$$\mathsf{a}_{\text{s.c.}} = \mathbb{C}[\boldsymbol{g}]\{\boldsymbol{q}\}\,, \tag{3.31}$$

which is the ring of arbitrary convergent power series in $\boldsymbol{q}$ and polynomials in $\boldsymbol{g}$. Note the difference with the decomposition (3.5) in the full quantum theory, which was over the polynomial ring $\mathsf{a} = \mathbb{C}[\boldsymbol{g}, \boldsymbol{q}]$ and not the power series ring $\mathbb{C}[\boldsymbol{g}]\{\boldsymbol{q}\}$. The polynomials $\tilde{P}_p$ are called the Weierstrass polynomials in the mathematical literature. Their roots are given by Puiseux expansions (power series expansions involving in general fractional powers of the instanton factors) that correspond to the small $\boldsymbol{q}$ expansions of the chiral operators expectation values. It is clear that if we perform analytic continuations along closed loops in parameter space that remain in the small $\boldsymbol{q}$ region (staying within the radius of convergence of the series defining the coefficients of the polynomials appearing in (3.30)), the polynomials $\tilde{P}_p$ remain invariant and thus the roots of two different factors in (3.30) cannot be smoothly connected. This explains the correspondence between (3.30) and (3.29). We also have a nice illustration of the importance of the base ring: going from the semi-classical approximation to the full quantum theory amounts to studying factorization properties over a polynomial ring instead of a power series ring. We shall present an explicit example in section 4.

3.9 Summary

Let us briefly recapitulate what we have done in the previous sections.

- The chiral sector of any supersymmetric gauge theory is described by a set of polynomial equations with coefficients in a ring of parameters a which in most cases is a simple polynomial ring, $\mathsf{a} = \mathbb{C}[\boldsymbol{g}, \boldsymbol{q}]$. In particular, if v is the number of vacua of the theory, any chiral operator satisfies a degree v algebraic equation with coefficients in a. The full set of operator constraints is always generated by a finite subset of equations.

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– The phases of the gauge theory can be studied by computing the decomposition of these polynomials in irreducible factors or more generally the prime decomposition of the ideal of operator relations.

– A given phase can always be described by a single "primitive" operator (which is not unique) that satisfies an irreducible polynomial equation. All the other operators are given by a polynomial expression in terms of the primitive operator.

In the next two sections we are going to apply these ideas to study two interesting models in details.

4. Application: Higgs and confinement

4.1 The model and the general theorem

We now focus on the $\mathrm{U}(N)$ model with N_{f} flavors (2.20) or more generally on

$$W_{\mathrm{tree}} = \mathrm{Tr}\, W(\phi) + \tilde{Q}^f m_f^{\ f'}(\phi) Q_{f'}\,, \tag{4.1}$$

with

$$W'(z) = \sum_{k=0}^{d} g_k z^k = g_d \prod_{k=1}^{d}(z - w_i)\,, \quad \det m_f^{\ f'}(z) = \prod_{k=1}^{N_{\mathrm{f}}}(z - m_k)\,. \tag{4.2}$$

The most general classical vacuum $|N_i;\nu_j\rangle_{\mathrm{cl}}$ is labeled by the numbers of eigenvalues of the matrix ϕ, $N_i \geq 0$ and $\nu_j = 0$ or 1, that are equal to w_i and m_j respectively [10]. The constraint

$$\sum_{i=1}^{d} N_i + \sum_{j=1}^{N_{\mathrm{f}}} \nu_j = N \tag{4.3}$$

must be satisfied. The gauge group $\mathrm{U}(N)$ is broken down to $\mathrm{U}(N_1) \times \cdots \times \mathrm{U}(N_d)$ in a vacuum $|N_i;\nu_j\rangle_{\mathrm{cl}}$. As explained in section 2.2.2, chiral symmetry breaking implies that the quantum vacua can be labeled as $|N_i, k_i;\nu_j\rangle$ with $k_i \in \mathbb{Z}_{N_i}$.

Definition 10. The *rank* r of a vacuum $|N_i, k_i;\nu_j\rangle$ is defined to be the number of non-zero integers N_i.

Taking into account the mass gap in the non-abelian unbroken factors of the gauge group, the low energy gauge group is $\mathrm{U}(1)^r$ and thus r counts the number of massless photons. This number cannot change when the parameters are smoothly varied and thus r is a phase invariant (this can also be trivially checked on the solution of the model). Let us note that for the model (4.2), $r \leq \min(N, d)$. The fundamental result conjectured in [10] that we want to prove can be summarized as follows.

Theorem 18. *The model* (4.1) *has, for a given value of the rank r, a unique phase $|r)$ containing all the vacua of rank r.*

This result is equivalent to the fact that one can always interpolate smoothly between two vacua $|N_i, k_i;\nu_j\rangle$ and $|N'_i, k'_i;\nu'_j\rangle$ that have the same value of r. It encompasses in particular all the possible interpolations between various "confining" and "Higgs" vacua.

4.2 Using the weak coupling approximation

4.2.1 Semi-classical phases

The proof of theorem 18 can be simplified if one realizes that many analytic continuations between vacua are trivial, in the sense that they can be described in the semi-classical regime by computing explicitly the expectation values in a semi-classical expansion. The associated irreducible polynomials can of course be written down straightforwardly, but this is cumbersome and useless in these cases. The algebraic method will be better used later to deal with the genuinely quantum interpolations, that cannot be understood semi-classically.

So let us compute the leading terms in a semi-classical expansion around an arbitrary vacuum $|N_i, k_i; \nu_j\rangle$. This expansion is governed by the gluino condensation in each unbroken $\mathrm{U}(N_i)$ factors of the gauge group. For example, the quantum effective superpotential is given by

$$W_{\mathrm{eff}}^{|N_i,k_i;\nu_j\rangle} = \sum_{i=1}^{d} N_i W(w_i) + \sum_{j=1}^{N_\mathrm{f}} \nu_j W(m_j) + \sum_{i=1}^{d} N_i \Lambda_i^3 e^{2i\pi k_i/N_i} + \cdots \tag{4.4}$$

where we have neglected subleading terms when $q \to 0$. The Λ_i are the dynamically generated scales for the unbroken gauge groups. In terms of the scale Λ of the $\mathrm{U}(N)$ gauge theory, which is itself related to the instanton factor by the relation (2.4)

$$q = \Lambda^{2N-N_\mathrm{f}}\,, \tag{4.5}$$

one has

$$\Lambda_i^{3N_i} = q\frac{W''(w_i)^{N_i}\prod_{j=1}^{N_\mathrm{f}}(w_i - m_j)}{\prod_{j\neq i}(w_i - w_j)^{2N_j}\prod_{j=1}^{N_\mathrm{f}}(w_i - m_j)^{2\nu_j}}\,\cdot \tag{4.6}$$

This formula is obtained by integrating out the various massive degrees of freedom: the denominator is produced by the W bosons charged under $\mathrm{U}(N_i)$ and the numerator comes from the massive matter fields, adjoint multiplet (term $W''(w_i)^{N_i}$) and fundamental flavors (term $\prod_{j=1}^{N_\mathrm{f}}(w_i - m_j)$).

The formulas (4.4) and (4.6) immediately show that:

- the vacua that have the same set of integers $\{N_i\}$ and $\{\nu_j\}$ can all be smoothly connected to each other. Indeed, arbitrary permutations of the N_i on the one hand and of the ν_j on the other hand can be obtained by performing an analytic continuation that induces the same permutations on the parameters w_i and m_j respectively. Note that under such an analytic continuation, the integers k_i do not change and remain associated with the same integers N_i.

- vacua corresponding to fixed values of the N_i and ν_j but arbitrary values of the k_i are all smoothly connected to each other by performing analytic continuations of the form $w_i - m_j \mapsto e^{2i\pi}(w_i - m_j)$.

This is all we can do at the semi-classical level. The semi-classical phase diagram (3.29) is thus made up of phases labeled by the set of integers $\{N_i\}$ and $\{\nu_j\}$ but it is impossible to interpolate between vacua that have different values of the N_i and the ν_j by staying at weak coupling.

Example 16. To understand clearly what we have done, let us consider for example the case of the $N = 2$, $N_{\mathrm{f}} = 3$ theory, with $d = 2$ in (4.2). This theory has fourteen vacua that can be labeled as $|N_1, k_1; N_2, k_2; \nu_1, \nu_2, \nu_3\rangle$. Three vacua have rank $r = 0$ ($|0,0;0,0;1,1,0\rangle$, $|0,0;0,0;1,0,1\rangle$, $|0,0;0,0;0,1,1\rangle$), ten vacua have rank $r = 1$ ($|2,0;0,0;0,0,0\rangle$, $|2,1;0,0;0,0,0\rangle$, $|0,0;2,0;0,0,0\rangle$, $|0,0;2,1;0,0,0\rangle$, $|1,0;0,0;1,0,0\rangle$, $|1,0;0,0;0,1,0\rangle$, $|1,0;0,0;0,0,1\rangle$, $|0,0;1,0;1,0,0\rangle$, $|0,0;1,0;0,1,0\rangle$, $|0,0;1,0;0,0,1\rangle$) and one vacuum has rank $r = 2$ ($|1,0;1,0;0,0,0\rangle$). From the semi-classical analysis only, we know that all the vacua of rank $r = 0$ are in the same phase. The vacuum at $r = 2$ yields another phase on its own. At rank $r = 1$, we have two distinct semiclassical phases, corresponding to either a U(2) unbroken gauge group (four "confining" vacua) or to a trivial U(1) unbroken gauge group (six "Higgs" vacua). Theorem 18 implies that, taking into account the strong coupling quantum effects, these ten vacua are actually in the same phase.

4.2.2 The strongly quantum problem

The semi-classical analysis of the previous subsection shows that the non-trivial interpolations correspond to changing the values of the non-zero integers N_i (and thus also of some of the ν_j according to (4.3)). This of course can be done step by step, and thus it is enough to show that any of the N_i can be changed by one unit as long as it remains non-zero. Since the scales (4.6) of the various $\mathrm{U}(N_i)$ factors can be separated at will, one can try to study this phenomenon in a limit where the theory reduces to a $\mathrm{U}(N_i)$ model of the form (2.20) with one flavor of quark (one flavor is enough to study changes of the number of colors by one unit). Precisely, if we choose for example $i = 1$, then we can consider the region of parameters where the W bosons and all the quarks except one are extremely massive, $w_j \to \infty$ and $m_j \to \infty$ for $j \geq 2$, while $W''(w_1) = \mu$ and the effective instanton factor

$$\frac{\prod_{j=2}^{N_{\mathrm{f}}}(w_1 - m_j)^{1-2\nu_j}}{\prod_{j=2}^{N}(w_1 - w_j)^{2N_j}}\, q = q_{\mathrm{eff}} \tag{4.7}$$

remains constant. Clearly, if the interpolation is possible in this limit, then it will be possible in the more general cases. Thus we see that the general theorem 18 can be derived from the following simplified lemma.

Lemma 19. *The model* (2.20) *with* $N_{\mathrm{f}} = 1$ *is realized in only one phase, i.e. the* N *"confining" vacua* $|N, k; 0\rangle$ *for* $0 \leq k \leq N-1$ *and the* $N-1$ *"Higgs" vacua* $|N-1, k; 1\rangle$ *for* $0 \leq k \leq N-2$ *can be smoothly connected to each other.*

This statement contains all the relevant strongly quantum information about the interpolation between Higgs and confining phases. It will be derived in 4.5 by proving that the glueball operator satisfies a degree $2N-1$ irreducible polynomial equation over $\mathbb{C}[\mu, m, q]$, where m is the mass of the flavor.

4.3 The operator relations

The chiral ring of the model (4.1) is generated by the operators

$$u_k = \operatorname{Tr}\phi^k\,, \quad v_k = -\frac{1}{16\pi^2}\operatorname{Tr} W^\alpha W_\alpha \phi^k\,, \quad t^{\ f}_{f',\,k} = \tilde{Q}^f \phi^k Q_{f'}\,. \tag{4.8}$$

As usual, it is useful to introduce the generating functions

$$R(z) = \sum_{k\geq 0}\frac{u_k}{z^{k+1}}\,, \quad S(z) = \sum_{k\geq 0}\frac{v_k}{z^{k+1}}\,, \quad G^{\ f}_{f'}(z) = \sum_{k\geq 0}\frac{t^{\ f}_{f',\,k}}{z^{k+1}}\,, \tag{4.9}$$

and also the function $F(z)$ defined by (2.74) that satisfies by construction

$$R(z) = \frac{F'(z)}{F(z)}\,\cdot \tag{4.10}$$

When $N_{\mathrm{f}} \geq N$ there are also baryonic operators, but they will play no rôle in our analysis. Indeed, it is enough to consider the operators (4.8) to prove that all the vacua at a given rank can be smoothly connected. From section 3.4 we then know that at a given rank the baryonic operators are simple polynomials in the generators (4.8).

When $N_{\mathrm{f}} < 2N$, the ring of parameters of the model is

$$\mathsf{a} = \mathbb{C}[g_0, \ldots, g_d, m_1, \ldots, m_{N_{\mathrm{f}}}, q]\,. \tag{4.11}$$

When $N_{\mathrm{f}} = 2N$ we must allow arbitrary series in q.

4.3.1 Kinematical and dynamical relations

We now need to write down a full set of operator relations. It is natural to distinguish "kinematical" and "dynamical" relations.

The kinematical relations come from the fact that the number of colors N in the theory is finite. Thus, amongst the generators (4.8), only the u_k for $1 \leq k \leq N$, the v_k and $t^{\ f}_{f',\,k}$ for $0 \leq k \leq N-1$ can be independent. As explained in 2.6, there is some freedom in defining the other operators. We choose to define the u_k for $k > N$ by imposing the constraint

$$F(z) + qU(z)/F(z) = P(z)\,, \tag{4.12}$$

where

$$U(z) = \prod_{f=1}^{N_{\mathrm{f}}}(z - m_f) \tag{4.13}$$

and $P(z)$ is a degree N polynomial. The condition (4.12) generalizes the choice (2.75) made in the case $N_{\mathrm{f}} = 0$. It is *equivalent* to relations of the form (2.72), where now the polynomials $\tilde{Q}_p$ also depend on the completely symmetric polynomials

$$\sigma_i = \sum_{f_1 < \cdots < f_i} m_{f_1}\cdots m_{f_i} \tag{4.14}$$

in the quark masses,

$$u_{N+p} = \tilde{Q}_p(u_1, \ldots, u_N; \sigma_1, \ldots, \sigma_{N_{\mathrm{f}}}; q)\,. \tag{4.15}$$

Similar kinematical constraints for the operators v_k and $t_{f',k}^{f}$ at $k \geq N$ also exist, but they don't need to be discussed independently. Indeed, it turns out that they follow from (4.12) and from the dynamical relations we now discuss.

The dynamical relations are the famous generalized Konishi anomaly equations. For our model, we have four infinite families of equations, labeled by an integer $n \geq -1$,

$$N\sum_{k\geq 0} g_k u_{n+k+1} + \sum_f t_{f,\,n+1}^{f} - 2\sum_{k_1+k_2=n} u_{k_1}v_{k_2} = 0 \tag{4.16}$$

$$N\sum_{k\geq 0} g_k v_{n+k+1} - \sum_{k_1+k_2=n} v_{k_1}v_{k_2} = 0 \tag{4.17}$$

$$N\bigl(t_{f'\,n+2}^{f} - m_f t_{f'\,n+1}^{f}\bigr) - v_{n+1}\delta_{f'}^{f} = 0 \tag{4.18}$$

$$N\bigl(t_{f'\,n+2}^{f} - m_{f'} t_{f'\,n+1}^{f}\bigr) - v_{n+1}\delta_{f'}^{f} = 0\,. \tag{4.19}$$

In terms of the generating functions (4.9), these equations read

$$NW'(z)R(z) + N\sum_f G_f^{f}(z) - 2S(z)R(z) = N^2\Delta_R(z) \tag{4.20}$$

$$NW'(z)S(z) - S(z)^2 = N^2\Delta_S(z) \tag{4.21}$$

$$N(z-m_f)G_{f'}^{f}(z) - S(z)\delta_{f'}^{f} = N\Delta_{f'}^{f}(z) \tag{4.22}$$

$$N(z-m_{f'})G_{f'}^{f}(z) - S(z)\delta_{f'}^{f} = N\tilde{\Delta}_{f'}^{f}(z)\,. \tag{4.23}$$

where the right hand side of the above equations are polynomials.

At the *perturbative* level, the equations (4.16)–(4.19) have been derived in [26]. In the perturbative approach, the kinematical relations are not given by (4.15), but by their classical counterpart obtained by setting $q = 0$. At the non-perturbative level, the anomaly equations get non-trivial quantum corrections. However, it turns out that these corrections can be made implicit for a privileged definition of the variables, which is precisely the one given by (4.12). A proof of this result in the case of the $N_{\mathrm{f}} = 0$ theory was given in [3] and the case of arbitrary N_{f} will appear in [27].

4.3.2 The ideal of operator relations

One approach to solve the model, used for example in [10], is to solve the anomaly equations, then to impose some ad hoc constraints on the generating functions, and finally to fix the remaining ambiguity by extremizing a postulated glueball superpotential. This approach is not appropriate in our framework, since we want to obtain a completely algebraic description of the solution.

We are going to show that both the ad hoc constraints imposed in [10] and the constraints coming from the glueball superpotential are automatically implemented when the relations (4.15) are taken into account in addition to the anomaly equations. Equivalently, the radical of the ideal generated by the relations (4.15)–(4.19) is the ideal $\mathscr{I}$ of operator relations defined in section 2.[9] Physically speaking, this means that the constraint (4.12)

[9] In all cases that we have checked explicitly using SINGULAR, the ideal generated by (4.15)–(4.19) is actually radical and thus coincides with $\mathscr{I}$. We believe that this is true in general but we have not tried to find a proof, since this result is not useful for our purposes.

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completely fixes the polynomials in the right hand side of (4.20)–(4.23), up to a discrete ambiguity corresponding to a choice of vacuum.

Let us focus on the model (2.20) with $W'(z) = \mu z$ since we know from the discussion in section 4.2 that the study of this case is sufficient for our purposes.[10] There is no difficulty in finding the general solution to (4.20)–(4.23) taking into account the asymptotic behaviour of the generating functions. First, by combining (4.22) and (4.23) and using the large z limit, we find that $t^{f}_{f',0}$ must be diagonal,

$$\langle \tilde{Q}^f Q_{f'} \rangle = t^{f}_{f',0} = t_f \delta^f_{f'} \, . \tag{4.24}$$

The generating functions are then expressed in terms of $v_0 = S$ and the t_f,

$$S(z) = \frac{N\mu}{2}\Big(z - \sqrt{z^2 - 4S/\mu}\Big) \tag{4.25}$$

$$G^{f}_{f'}(z) = \delta^f_{f'} \frac{\frac{1}{N}S(z) + t_f}{z - m_f} \, , \tag{4.26}$$

$$R(z) = \frac{1}{2}\sum_f \frac{1}{z - m_f} + \frac{1}{\sqrt{z^2 - 4S/\mu}}\bigg(N - \frac{1}{2}\sum_f \frac{z + 2t_f/\mu}{z - m_f}\bigg) . \tag{4.27}$$

By expanding at large z, we see that the formulas (4.25)–(4.27) are equivalent to identities giving the infinite number of operators in (4.8) in terms of polynomials in S and the t_f with coefficients in $\mathsf{k} = \mathbb{C}(q, \mu, m_1, \ldots, m_{N_\mathrm{f}})$ (the instanton factor q actually does not enter into these relations). We can thus write the chiral ring as the quotient ring

$$\mathbb{A} = \mathsf{k}[t_1, \ldots, t_{N_\mathrm{f}}, S]/\mathscr{I} \, , \tag{4.28}$$

where $\mathscr{I}$ is now the ideal generated by the set of operator relations between the generators S and t_f. This ideal contains all the non-trivial quantum information.

The ideal $\mathscr{I}$ can be computed in principle as follows. From (4.27), we find polynomial relations of the form

$$u_k = \rho_{u_k}(t_1, \ldots, t_f, S) \tag{4.29}$$

with $\rho_{u_k} \in \mathsf{k}[X_1, \ldots, X_{N_\mathrm{f}+1}]$. By plugging (4.29) into (4.15), we find in principle an infinite set of constraints on the generators S and t_f. By the noetherian property, we know that only a finite number of these constraints are independent. It is not difficult to use this method to study simple cases (in practice it turns out that the first $N_\mathrm{f} + 1$ non-trivial equations generate $\mathscr{I}$), but it becomes quite cumbersome for large values of N and N_f, in particular because the polynomials in (4.29) and (4.15) are quite complicated. Fortunately, it is possible to find a much simpler set of generators for the ideal $\mathscr{I}$.

4.3.3 Simplifying the relations

The generating function $R(z)$ given in (4.27) is a two-sheeted analytic function which has generically $2N_\mathrm{f}$ poles located at $z = m_f$ on both sheets. On the other hand, (4.12) can be

[10] The general case can be treated along the same lines, see also [21].

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solved explicitly and from (4.10) we obtain an alternative formula for $R(z)$,

$$R(z) = \frac{1}{2}\sum_f \frac{1}{z-m_f} + \frac{1}{\sqrt{P(z)^2 - 4qU(z)}}\left(P'(z) - \frac{1}{2}P(z)\sum_f \frac{1}{z-m_f}\right). \tag{4.30}$$

From this formula, we see that $R(z)$ has poles only at $z = m_f$, consistently with (4.27), but we find an additional condition: because $U(m_f) = 0$, the pole is either on the first sheet or on the second sheet (depending on the sign of the square root) but not on both. Let us note that this condition has been imposed in an ad hoc way in the literature [10]. In our framework, it is essential to understand that it follows from the algebraic relations (4.15), and that no additional ad hoc conditions need to be imposed.

The total number of poles of $R(z)$ is thus N_{f} and not $2N_{\mathrm{f}}$. This yields N_{f} constraints on (4.27) and thus on the t_f and S. A pole at $z = m_f$ on the first sheet (the first sheet is defined by the condition $R(z) \sim N/z$ at infinity) corresponds to a vacuum with $\nu_f = 1$, while a pole at $z = m_f$ on the second sheet corresponds to a vacuum with $\nu_f = 0$. The residues of the poles at $z = m_f$ can be computed from (4.27) and are given by

$$\frac{1}{2}\left(1 \mp \frac{m_f + 2t_f/\mu}{\sqrt{m_f^2 - 4S/\mu}}\right), \tag{4.31}$$

with the minus or plus sign corresponding to the poles on the first and second sheets respectively. The fact that one of these residues must vanish is thus equivalent to $(m_f + 2t_f/\mu)^2 = m_f^2 - 4S/\mu$ or

$$t_f^2 + \mu m_f t_f + \mu S = 0\,. \tag{4.32}$$

This yields N_{f} algebraic equations that automatically belong to the ideal $\mathscr{I}$ in (4.28). As we have explained, these equations are consequences of (4.29), but are much simpler and easier to use.

We need one additional equation (at least) to find a full set of generators of $\mathscr{I}$. This last equation determines the glueball S. In the Dijkgraaf-Vafa matrix model approach, it is found by extremizing the glueball superpotential. In our approach, we simply need to use one non-trivial (i.e. q-dependent) relation of the form (4.15). If we expand $F(z)$ defined in (2.74) as

$$F(z) = z^N - \sum_{k\geq 1} F_k z^{N-k}\,, \tag{4.33}$$

the simplest relation that follows from (4.12) is simply

$$F_{2N-N_{\mathrm{f}}} = q\,. \tag{4.34}$$

Equations (4.32) and (4.34) are in principle all we need. The claim is that they generate the ideal $\mathscr{I}$ and that this ideal is prime for $N_{\mathrm{f}} < N$ (meaning that there is only one phase is this case) or has two components in the prime decomposition (3.14) when $N_{\mathrm{f}} \geq N$ (because in this case we have a phase with no quantum correction corresponding to a completely broken gauge group). If we eliminate the variables t_f from (4.32) and (4.34), we should find a polynomial equation for S whose degree is equal to the number v of quantum vacua computed in section 2.2.2. If $N_{\rm f} < N$, this polynomial should be irreducible and if $N_{\rm f} \geq N$ it should have two irreducible components. We shall prove all these properties in full generality in 4.5, by simplifying further the set of generators of the ideal $\mathscr{I}$. In particular, we shall be able to find an explicit formula for the polynomial equation satisfied by S. However, before we tackle the general case, let us first study a simple illustrative example.

4.4 A simple case in details

Let us look at the theory with $N = 2$ and $N_{\rm f} = 1$. It is the simplest non-trivial example, yet it displays all the important qualitative features that are also found in the most general situation. The model has three quantum vacua, two "confining" $|2,0;0\rangle = |\mathrm{C},1\rangle$ and $|2,1;0\rangle = |\mathrm{C},2\rangle$ with unbroken gauge group and chiral symmetry breaking and one "Higgs" $|1,0;1\rangle = |\mathrm{H}\rangle$. Our main goal is to show that these three vacua are in the same phase.

We have to implement eq. (4.34) which here reads $F_3 = q$. Expanding (2.74), it is straightforward to find

$$F_3 = \frac{1}{3}u_3 - \frac{1}{2}u_1 u_2 + \frac{1}{6}u_1^3 = q\,. \tag{4.35}$$

Expanding (4.27), we also find

$$u_1 = -t/\mu\,,\ u_2 = (3S - mt)/\mu\,,\ u_3 = -(2St + \mu m S + \mu m^2 t)/\mu^2 \tag{4.36}$$

where we have noted $m_1 = m$ and $t_1 = t = \tilde{Q}Q$ is the meson operator. Plugging (4.36) into (4.35) and also taking into account (4.32), we find the two relations that generate the ideal $\mathscr{I}$,

$$t^2 + \mu m t + \mu S = 0 \tag{4.37}$$

$$t^3 + \mu(3mt - 5S)t + 2\mu^2 m(mt + S) + 6\mu^3 q = 0\,. \tag{4.38}$$

We can now illustrate explicitly many properties discussed in sections 2 and 3. We are going to check successively that:

(i) S and t both satisfy irreducible degree three polynomial equations $P_S = 0$ and $P_t = 0$ over $\mathbb{C}[\mu, m, q]$. This will imply immediately that the Higgs and the two confining vacua belong to the same phase.

(ii) S and t are primitive operators and thus all the operators in the theory can be written as polynomials in S or in t with coefficients in $\mathbb{C}(\mu, m, q)$.

(iii) At weak coupling, the Higgs and confining vacua are not connected. This means that P_S and P_t actually factorize over $\mathbb{C}[\mu, m]\{q\}$.

Point (i) can be checked by eliminating S or t from the two equations (4.37) and (4.38). It is trivial to eliminate S using (4.37) and plugging the result into (4.38) we find the polynomial equation for t,

$$P_t(t) = t^3 + \mu m t^2 + \mu^3 q = 0\,. \tag{4.39}$$

To find the equation for S, we first eliminate t^3 from (4.38) by multiplying (4.37) by t and subtracting, and then we eliminate t^2 from the resulting equations by using the same procedure. This yields

$$St = \mu^2 q \tag{4.40}$$

and

$$P_S(S) = S^3 + \mu^2 mqS + \mu^3 q^2 = 0\,. \tag{4.41}$$

Let us now show that P_t is irreducible. We write

$$P_t(t,\mu,m,q) = A(t,\mu,m,q)B(t,\mu,m,q)\,. \tag{4.42}$$

Since the degree in q of P_t is one, either A or B (let us say A) must be independent of q. By setting $q = 0$ in (4.42) we thus find

$$t^2(t+\mu m) = A(t,\mu,m)B(t,\mu,m,q=0)\,. \tag{4.43}$$

But A cannot be a multiple of t or of $t+\mu m$: it would contradict (4.42) since $P_t(t = 0,\mu,m,q) \neq 0$ and $P_t(t=-\mu m,\mu,m,q) \neq 0$. Thus (4.43) implies that A doesn't depend on t, proving that P_t is irreducible. The birational equivalence (4.40) between the two equations (4.39) and (4.41) also immediately implies that P_S is irreducible as well. *This proves that the confining and Higgs vacua are in the same phase.*

Since the polynomial equations satisfied by S and t are irreducible, they both must be primitive operators. From the discussion in section 3.4, we know that all the operators of the theory can then be expressed as polynomials in either t or S. We can now see this explicitly. From (4.25)–(4.27), it is manifest that all the operators (4.8) are polynomials in t and S. These immediately yield polynomials in t, since eq. (4.37) shows that S itself is a polynomial in t. They also yield polynomials in S, since we can also express t as a polynomial in S by using (4.40) and (4.41),

$$t = -\mu m - \frac{S^2}{\mu q}\,\cdot \tag{4.44}$$

Let us finally illustrate the relation between the weak coupling expansion and the full quantum theory, using for example the glueball superfield S. It is not difficult to solve (4.41) at small q. The three roots, corresponding to the expectation values in the three vacua, have series expansion of the form

$$\langle \mathrm{H}|S|\mathrm{H}\rangle = \mu m^2 \sum_{k\geq 1} h_k (q/m^3)^k \tag{4.45}$$

$$\langle \mathrm{C},1|S|\mathrm{C},1\rangle = \mu m^2 \sum_{k\geq 1} c_k (q/m^3)^{k/2} \tag{4.46}$$

$$\langle \mathrm{C},2|S|\mathrm{C},2\rangle = \mu m^2 \sum_{k\geq 1} (-1)^k c_k (q/m^3)^{k/2}\,. \tag{4.47}$$

The numerical coefficients h_k, c_k can be easily computed, for example

$$h_1 = -1\,,\ h_2 = 1\,,\ h_3 = -3\,,\ c_1 = i\,,\ c_2 = 1/2\,,\ c_3 = 3i/8\,,\ldots \tag{4.48}$$

The series expansions (4.45)–(4.47) clearly show that the vacua $|\mathrm{C},1\rangle$ and $|\mathrm{C},2\rangle$ can be analytically continued into each other at small q, but that they are disconnected from the Higgs vacuum in this approximation. Algebraically, the polynomial P_S factorizes,

$$P_S = \tilde{P}_S^{|\mathrm{C}\rangle}\tilde{P}_S^{|\mathrm{H}\rangle}\,, \tag{4.49}$$

where

$$\tilde{P}_S^{|\mathrm{C}\rangle}(S) = \bigl(S - \langle \mathrm{C},1|S|\mathrm{C},1\rangle\bigr)\bigl(S - \langle \mathrm{C},2|S|\mathrm{C},2\rangle\bigr)\,,\ \tilde{P}^{|\mathrm{H}\rangle}(S) = S - \langle \mathrm{H}|S|\mathrm{H}\rangle \tag{4.50}$$

are the Weierstrass polynomials discussed in 3.8 whose coefficients are arbitrary series in q, i.e. elements of $\mathbb{C}[\mu,m]\{q\}$. Going from the weak coupling approximation to the full quantum theory is mathematically equivalent to allowing only polynomials in q, and not arbitrary series, for the coefficients of the polynomial. As we have already shown, a non-trivial decomposition of the form (4.49) is then no longer possible: P_S is irreducible over $\mathbb{C}[\mu,m,q]$, showing that strong coupling effects make the Higgs and confining phases indistinguishable.

4.5 The general case

As explained at the end of section 2.2.2, the model (2.20) that we are studying has vacua of rank one and also vacua of rank zero when $N_\mathrm{f}\geq N$. These vacua of rank zero are trivial in the sense that they have no quantum correction. They correspond to a trivial solution of (4.12) and (4.20)–(4.23) for which $S(z)=0$ and $F(z)=\prod_{i=1}^{N}(z-m_{f_i})$ is a polynomial dividing $U(z)$. The $v_0=\binom{N_\mathrm{f}}{N}$ rank zero vacua can trivially be connected to each other by permuting the masses m_f. The ideal of operator relations thus decomposes as

$$\mathscr{I} = \mathscr{I}_{|0)}\cap\mathscr{I}_{|1)}\,, \tag{4.51}$$

where $\mathscr{I}_{|0)}$ is the prime ideal of classical relations at rank zero. All the non-trivial quantum information is included in the operator relations in the vacua of rank one $\mathscr{I}_{|1)}$. Moreover, note that when $N_\mathrm{f}<N$ there is no vacuum of rank zero and $\mathscr{I}=\mathscr{I}_{|1)}$. Thus in all cases, the theorem 18 that we want to prove is equivalent to the fact that $\mathscr{I}_{|1)}$ is prime.

4.5.1 Simple generators for $\mathscr{I}_{|1)}$

Using (4.34) for general N and N_f is not very convenient. To find the general form of the algebraic equation we need, the best approach is to solve directly the constraint (4.12). Moreover, as explained above, we can focus on the ideal $\mathscr{I}_{|1)}$.

First, it will be useful, in an intermediate stage, to solve explicitly (4.32) as

$$t_f = -\frac{\mu}{2}\Bigl(m_f + (2\nu_f-1)\sqrt{m_f^2-4S/\mu}\Bigr)\,. \tag{4.52}$$

The integers $\nu_f=0$ or 1 correspond to the labels introduced in 4.1 to distinguish the various vacua. From (4.10) and (4.27) it is then straightforward to obtain, by direct integration, an explicit expression for $F(z)$. Using

$$\int_\infty^z \mathrm{d}x\Big(\frac{1}{\sqrt{x^2-a^2}}-\frac{1}{x}\Big)=\ln\frac{z+\sqrt{z^2-a^2}}{2z} \tag{4.53}$$

$$\int_\infty^z \frac{\mathrm{d}x}{(x-m)\sqrt{x^2-a^2}}=\frac{1}{\sqrt{m^2-a^2}}\ln\frac{(z-m)(m+\sqrt{m^2-a^2})}{mz-a^2+\sqrt{(m^2-a^2)(z^2-a^2)}}\,, \tag{4.54}$$

we get

$$F(z)=\left(\frac{z+\sqrt{z^2-4S/\mu}}{2}\right)^{N-N_\mathrm{f}/2}\prod_f(z-m_f)^{\nu_f}$$
$$\prod_f\left(\frac{m_f+\sqrt{m_f^2-4S/\mu}}{m_fz-4s/\mu+\sqrt{(m_f^2-4S/\mu)(z^2-4S/\mu)}}\right)^{\nu_f-1/2}. \tag{4.55}$$

Let us now perform an analytic continuation, starting from the sheet where $F(z)\sim z^N$ at infinity and going through the cut of the square root $\sqrt{z^2-4S/\mu}$. Here we assume that $S\neq 0$, i.e. that the cut is non-trivial. This means that we exclude the trivial classical solutions $S=0$ or in other words that we are looking for operator relations in $\mathscr{I}_{|1)}$. The analytic continuation produces the changes

$$\sqrt{z+\sqrt{z^2-4S/\mu}}\ \longrightarrow\ \sqrt{z-\sqrt{z^2-4S/\mu}} \tag{4.56}$$

$$\sqrt{m_fz-4s/\mu+\sqrt{(m_f^2-4S/\mu)(z^2-4S/\mu)}}\ \longrightarrow$$
$$-\sqrt{m_fz-4s/\mu-\sqrt{(m_f^2-4S/\mu)(z^2-4S/\mu)}} \tag{4.57}$$

The global minus sign in (4.57) comes from crossing part of the double cut that originates from the double zero of $m_fz-4s/\mu-\sqrt{(m_f^2-4S/\mu)(z^2-4S/\mu)}$ at $z=m_f$. The function F thus becomes

$$F(z)\ \longrightarrow\ \hat F(z)=\left(\frac{z-\sqrt{z^2-4S/\mu}}{2}\right)^{N-N_\mathrm{f}/2}\prod_f(z-m_f)^{\nu_f}$$
$$(-1)^{N_\mathrm{f}}\prod_f\left(\frac{m_f+\sqrt{m_f^2-4S/\mu}}{m_fz-4s/\mu-\sqrt{(m_f^2-4S/\mu)(z^2-4S/\mu)}}\right)^{\nu_f-1/2}. \tag{4.58}$$

On the other hand, (4.12) implies that

$$\hat F(z)=qU(z)/F(z)\,. \tag{4.59}$$

Comparing (4.58) and (4.59), using a few simple algebraic manipulations including the identity

$$\Bigl(m_f+\sqrt{m_f^2-4S/\mu}\Bigr)^{1-2\nu_f}=\Bigl(\frac{\mu}{4S}\Bigr)^{\nu_f}\Bigl(m_f+(1-2\nu_f)\sqrt{m_f^2-4S/\mu}\Bigr) \tag{4.60}$$

$$=-\Bigl(\frac{\mu}{4S}\Bigr)^{\nu_f}\frac{2S}{t_f}\,, \tag{4.61}$$

we obtain a necessary and sufficient condition for (4.27) and (4.12) to be simultaneously satisfied,

$$S^{N-N_\mathrm{f}}\prod_{f=1}^{N_\mathrm{f}}t_f=\mu^N q\,. \tag{4.62}$$

This equation generalizes (4.40) to arbitrary N and N_f. Together with (4.32), we have obtained a simple set of generators for the ideal $\mathscr{I}_{|1)}$ of operator relations,

$$\mathscr{I}_{|1)}=\left(t_1^2+\mu m_1t_1+\mu S,\ldots,t_{N_\mathrm{f}}^2+\mu m_{N_\mathrm{f}}t_{N_\mathrm{f}}+\mu S,S^{N-N_\mathrm{f}}\prod_{f=1}^{N_\mathrm{f}}t_f-\mu^N q\right). \tag{4.63}$$

4.5.2 The polynomial equations for S

From (4.51), we know that the polynomial P_S for the glueball S is of the form

$$P_S(S)=S^{\binom{N_\mathrm{f}}{N}}P_S^{|1)}(S)\,, \tag{4.64}$$

where conventionally we set $\binom{N_\mathrm{f}}{N}=0$ if $N_\mathrm{f}<N$. The polynomial $P_S^{|1)}$ must be of degree v_1 given by (2.24). It can be constructed in principle by eliminating the variables t_f from the relations defining $\mathscr{I}_{|1)}$.

This is extremely elementary when $N_\mathrm{f}=1$. In this case, noting $m_1=m$ and $t_1=t$, the relations are simply

$$t^2+\mu mt+\mu S=0 \tag{4.65}$$

$$S^{N-1}t-\mu^N q=0\,. \tag{4.66}$$

Solving (4.66) for t and plugging the result in (4.65) we find

$$P_S(S)=S^{2N-1}+\mu^N mqS^{N-1}+\mu^{2N-1}q^2\quad\text{for }N_\mathrm{f}=1\,. \tag{4.67}$$

This equation generalizes (4.41) to arbitrary N. The case $N_\mathrm{f}=2$ is a little bit more tedious but the calculation is still tractable and yields

$$P_S^{|1)}(S)=S^{4N-4}-\mu^N m_1m_2qS^{3N-4}-2\mu^{2N-2}q^2S^{2N-2}+\mu^{10N-1}(m_1^2+m_2^2)q^2S^{2N-3}$$
$$-\mu^{3N-2}m_1m_2q^3S^{N-2}+\mu^{4N-4}q^4\quad\text{for }N_\mathrm{f}=2\,. \tag{4.68}$$

For $N_\mathrm{f} \geq 3$ the calculations become daunting. In particular, the degree of $P_S^{|1)}$ grows exponentially. As a last example, we indicate the solution for $N = 2$ and $N_\mathrm{f} = 3$,

$$\begin{aligned} P_S^{|1)} =& S^5 + 4\mu q^2 S^4 + \mu^2\big[6q^3 + m_1m_2m_3 - 2(m_1^2+m_2^2+m_3^2)q\big]qS^3 \\ &+ \mu^3\big[4q^4 - 5m_1m_2m_3q + m_1^2m_2^2 + m_1^2m_3^2 + m_2^2m_3^2 - 4(m_1^2+m_2^2+m_3^2)q^2\big]q^2S^2 \\ &+ \mu^4\big[q^5 - 5m_1m_2m_3q^2 - 2(m_1^2+m_2^2+m_3^2)q^3 + m_1m_2m_3(m_1^2+m_2^2+m_3^2) \\ &+ (m_1^4+m_2^4+m_3^4)q\big]q^3S + \mu^5\big[m_1m_2m_3q^3 + m_1^2m_2^2m_3^2 + m_1m_2m_3(m_1^2+m_2^2+m_3^2)q \\ &+ (m_1^2m_2^2 + m_1^2m_3^2 + m_2^2m_3^2)q^2\big]q^4 \quad \text{for } N=2 \text{ and } N_\mathrm{f}=3\,. \end{aligned} \tag{4.69}$$

Interestingly, it is actually possible to give a general formula for $P_S^{|1)}$. We claim that

$$P_S^{|1)}(S) = \prod_{f=1}^{N_\mathrm{f}} \prod_{\nu_f=0}^{1} \left[S^{N-N_\mathrm{f}/2} - \frac{\mu^N q}{(4S)^{N_\mathrm{f}/2}} \prod_{f'=1}^{N_\mathrm{f}} \Big(-m_{f'} + (2\nu_{f'}-1)\sqrt{m_{f'}^2 - 4S/\mu} \Big) \right] \tag{4.70}$$

for $N_\mathrm{f} \leq N$. From (4.52) and (4.62) it is clear that $P_S^{|1)}(S) = 0$. The formula is single-valued by construction and thus, by an argument already used many times, we know that the right hand side of (4.70) must be a rational function. This means that when we expand (4.70), all the square roots automatically cancel. Actually, we have chosen the powers of S in (4.70) such that, for $N_\mathrm{f} \leq N$, only positive powers of S enter in $P_S^{|1)}$, with

$$P_S^{|1)}(S) = S^{(2N-N_\mathrm{f})2^{N_\mathrm{f}-1}} + \cdots + \mu^{(2N-N_\mathrm{f})2^{N_\mathrm{f}-1}} q^{2^{N_\mathrm{f}}}\,. \tag{4.71}$$

When $N_\mathrm{f} > N$, the small S behaviour is no longer necessarily dominated by the second terms in the bracket in (4.70) and there are thus negative powers of S in (4.70). It is not difficult to see that by multiplying by a suitable power of S we obtain a polynomial with the correct degree (2.24). For example, one can derive eq. (4.69) most efficiently using this method.

4.5.3 The irreducibility of $P_S^{|1)}$

Let us finally prove that the ideal (4.63) is prime. From the analysis in section 4.2, we know that if the ideal is prime in the case $N_\mathrm{f} = 1$, it will automatically be prime for all values of N_f.

We thus consider the degree $2N-1$ polynomial (4.67). To prove the irreducibility, we can proceed for example as in 4.4 below eq. 4.42. Let us assume that

$$P_S(S,\mu,m,q) = A(S,\mu,m,q)B(S,\mu,m,q) \tag{4.72}$$

where A and B are polynomials in S with coefficients in $\mathbb{C}[\mu,m,q]$. Assume that A and B both depend on q. Then their degree in q must be one. This is possible if and only if the roots of P_S, viewed as a degree two polynomial in q, are rational functions of S, μ and m. But this is not so, because the discriminant

$$\Delta = \mu^{2N-1}S^{2N-2}(m^2\mu - 4S) \tag{4.73}$$

is not a perfect square. We can thus assume that A, for example, is independent of q. Eq. (4.72) for $q = 0$ then implies that $S^{2N-1} = A(S, \mu, m)B(S, \mu, m, q = 0)$ and thus $A(S, \mu, m) = S^p \tilde{A}(\mu, m)$ for some $p \geq 0$. But $P_S(S = 0) \neq 0$ and thus necessarily $p = 0$ and A does not depend on S. This completes the proof: *there is no distinction between Higgs and confining vacua in our theory.*

The above reasoning also shows that S is a primitive operator in the case $N_{\mathrm{f}} = 1$. Actually, from the small q expansion and using proposition 17, it is very simple to show that S is a primitive operator for all N_{f}. In particular, proposition 16 then implies that $P_S^{|1)}$ given by (4.70) is irreducible for all N_{f}, a rather non-trivial algebraic result.

5. On the phases of the theory with one adjoint

We now focus on the theory with only one adjoint chiral superfield (2.10). When only adjoint fields are present, the screening mechanism, which is responsible for the equivalence between Higgs and confinement in theories with fundamentals, cannot occur. As a result, the phase structure of the model is much more intricate [7, 8, 5].

We are going to use the algebraic techniques introduced in the previous sections coupled with the computer algebra systems Singular and PHC [13, 14] to compute the full phase diagram for all gauge groups $\mathrm{U}(N)$ with $2 \leq N \leq 7$ (the cases $2 \leq N \leq 4$ were already worked out in [7, 8] and some phases at $N = 5$ and $N = 6$ were also discussed in [8, 5]). One of our goal is to present several non-trivial examples of irreducible polynomial equations satisfied by primitive operators.

5.1 The operator relations

The chiral ring is generated by the operators

$$u_k = \operatorname{Tr} \phi^k \,, \quad v_k = -\frac{1}{16\pi^2} \operatorname{Tr} W^\alpha W_\alpha \phi^k \,. \tag{5.1}$$

As in 4.3, we introduce the generating functions

$$R(z) = \sum_{k \geq 0} \frac{u_k}{z^{k+1}} = \frac{F'(z)}{F(z)} \,, \quad S(z) = \sum_{k \geq 0} \frac{v_k}{z^{k+1}} \,. \tag{5.2}$$

The field of parameters of the model is given by

$$\mathsf{k} = \mathbb{C}(g_0, \ldots, g_d, q) \,, \tag{5.3}$$

where d is the degree of the derivative $W'(\phi)$ of the tree-level superpotential. We shall always assume that $d \leq N$, since higher values of the degree do not yield new phases.

As in 4.3, we have kinematical and dynamical operator relations. We have already studied the kinematical relations in section 2.6, example 9. They are of the form (2.72) and are equivalent to the constraint (2.75). The dynamical relations, on the other hand, are special cases of (4.16) and (4.17) in which the fundamentals are integrated out. The

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full set of relations thus read

$$u_{N+p} = \tilde{Q}_p(u_1, \dots, u_N; q) \tag{5.4}$$

$$N \sum_{k=0}^{d} g_k u_{n+k+1} - 2 \sum_{k_1+k_2=n} u_{k_1} v_{k_2} = 0 \tag{5.5}$$

$$N \sum_{k=0}^{d} g_k v_{n+k+1} - \sum_{k_1+k_2=n} v_{k_1} v_{k_2} = 0 \tag{5.6}$$

for any $p \geq 1$ and $n \geq -1$, or equivalently in terms of the generating functions

$$F(z) + q/F(z) = P(z) \tag{5.7}$$

$$NW'(z)R(z) - 2R(z)S(z) = N^2 \Delta_R(z) \tag{5.8}$$

$$NW'(z)S(z) - S(z)^2 = N^2 \Delta_S(z) \tag{5.9}$$

where P, Δ_R and Δ_S are polynomials. Eq. (5.6) can be solved to express all the v_k for $k \geq d$ in terms of $v_0, \dots, v_{d-1}$. Eq. (5.5) can then be used to express all the u_k for $k \geq d$ in terms of $u_1, \dots, u_{d-1}$ and $v_0, \dots, v_{d-1}$. This can be made explicit by solving (5.8) and (5.9),

$$S(z) = \frac{N}{2}\Big(W'(z) - \sqrt{W'(z)^2 - 4\Delta_S(z)}\Big) \tag{5.10}$$

$$R(z) = \frac{N\Delta_R(z)}{\sqrt{W'(z)^2 - 4\Delta_S(z)}} \cdot \tag{5.11}$$

The above formulas give all the operators u_k and v_k in terms of the coefficients of the polynomials

$$\Delta_R(z) = g_d z^{d-1} + \sum_{k=0}^{d-2} a_k z^k \tag{5.12}$$

$$\Delta_S(z) = \sum_{k=0}^{d-1} b_k z^k \,. \tag{5.13}$$

There is a simple linear mapping betweem the coefficients $a_0, \dots, a_{d-2}, b_0, \dots, b_{d-1}$ and the operators $u_1, \dots, u_{d-1}, v_0, \dots, v_{d-1}$ given by

$$\Delta_R(z) = \frac{1}{N} \operatorname{Tr} \frac{W'(z) - W'(\phi)}{z - \phi} \,, \quad \Delta_S(z) = -\frac{1}{16\pi^2 N} \operatorname{Tr} W^\alpha W_\alpha \frac{W'(z) - W'(\phi)}{z - \phi} \cdot \tag{5.14}$$

The chiral ring can thus be expressed as

$$\mathbb{A} = \mathsf{k}[a_0, \dots, a_{d-2}, b_0, \dots, b_{d-1}]/\mathscr{I} \,, \tag{5.15}$$

where the ideal $\mathscr{I}$ is generated by the relations obtained by using (5.4). From the noetherian property, we know that only a finite number of relations is required. Indeed, we have the following simple lemma.

Lemma 20. *The ideal* $\mathscr{I}$ *in* (5.15) *is generated by the relations* (5.4) *for* $1 \leq p \leq N+2d-2$.

Indeed, the hypothesis of the lemma is equivalent to the condition

$$F(z) + q/F(z) = P(z) + \mathcal{O}(z^{-N-2d+1})\,. \tag{5.16}$$

Using $R = F'/F$ and (5.11), this yields

$$R(z) = \frac{P'(z)}{\sqrt{P(z)^2 - 4q}} + \mathcal{O}(z^{-2N-2d}) = \frac{N\Delta_R(z)}{\sqrt{W'(z)^2 - 4\Delta_S(z)}}\,. \tag{5.17}$$

Squaring this equality and multiplying by the denominators we find

$$P'(z)^2\bigl(W'(z)^2 - 4\Delta_S(z)\bigr) - N^2\Delta_R^2(z)\bigl(P(z)^2 - 4q\bigr) = \mathcal{O}(z^{-1})\,. \tag{5.18}$$

Since the left hand side of this equality is a polynomial, it must identically vanish. Working backward and using the asymptotics at infinity $R(z) \sim N/z$, we deduce that (2.77) and thus by integration (2.76) are valid. Equivalently, the full set of equations (5.4) follows.

5.2 Methodology

5.2.1 SINGULAR and PHC

SINGULAR [13] is a symbolic computer software for commutative algebra and algebraic geometry. It implements rigorous and powerful algorithms that can compute, amongst many other things, the primary decomposition (3.14). In principle, we can put the explicit formulas for the generators of the ideal $\mathscr{I}$ given by the lemma 20 in SINGULAR and obtain as the output the full phase diagram with explicit formulas for the generators of the operator relations in each phase. Using the same algorithms, SINGULAR can also factorize complicated polynomials and we have used it heavily below to prove the irreducibility of our polynomial equations.

PHC is a numerical software for algebraic geometry that can also compute (with a certain degree of certainty) the decomposition of an affine variety into irreducible components. The algorithms in PHC (which means Polynomial Homotopy Continuation) are very much in line with the analytic approach to compute the phase diagram, section 3.1. The software computes the intersection points (called "witness points") of the variety under study with generic hyperplanes and study the permutations that these points undergo when the hyperplanes are moved randomly. The orbits of the permutation group acting on the witness points yield the irreducible components of the variety. One loophole is that one can never be sure to obtain all the possible permutations between the witness points, since the number of random loops in hyperplane space that the computer can sample is always finite. Nevertheless, the program can be used with confidence to prove the irreducibility of a given component, by finding enough permutations to ensure that the action of the permutation group is transitive.

The simultaneous use of both PHC and SINGULAR can be quite effective. In particular, it occurs frequently that one programme is much more efficient in terms of CPU time than the other, depending on the details of the particular case under study. However, because only SINGULAR provides fully rigorous results, we have actually double-checked all our calculations in the present paper using both softwares.

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5.2.2 Some phase invariants

There exists a few simple quantities that must be phase invariants [5]. These invariants are very useful and simplify the computation of the phase diagram.

The rank. The formulas (2.77) and (5.11) are compatible only if the following standard factorization equations are satisfied,

$$P(z)^2 - 4q = M_{N-r}(z)^2 C_{2r}(z) \tag{5.19}$$

$$W'(z)^2 - 4\Delta_S(z) = g_d^2 N_{d-r}(z)^2 C_{2r}(z)\,, \tag{5.20}$$

where r is an integer satisfying $1 \leq r \leq \min(d, N) = d$ and M_{N-r}, N_{d-r} and C_{2r} are monic polynomials of degree $N-r$, $d-r$ and $2r$ respectively. These conditions show that the generating functions R and F defined in (5.2) and (2.74) are both single valued on the genus $r-1$ hyperelliptic curve

$$\mathcal{C}_r : \; y^2 = C_{2r}(z)\,. \tag{5.21}$$

Clearly, the integer r cannot change by analytic continuation and thus it is a phase invariant. By looking at the classical limit, it is straightforward to check that r corresponds to the rank of the vacua, defined in section 2.2.2, example 3.

A refinement of the rank. Let us note that the polynomials

$$P_\pm(z) = P(z) \mp 2q^{1/2} \tag{5.22}$$

cannot have common roots. Since $P^2 - 4q = P_+ P_-$, (5.19) implies that

$$P_\pm(z) = M_\pm(z)^2 C_\pm(z) \tag{5.23}$$

where $M_\pm$ and $C_\pm$ are polynomials of degrees $s_\pm$ and $N - 2s_\pm$ respectively, with

$$s_+ + s_- = N - r\,, \quad s_\pm \leq N/2\,, \tag{5.24}$$

and

$$M_{N-r} = M_+ M_-\,, \quad C_{2r} = C_+ C_-\,. \tag{5.25}$$

When $q \mapsto e^{2i\pi} q$, the integers s_+ and s_- are permuted, but clearly the unordered set of integers $\{s_+, s_-\} = \{s_-, s_+\}$ cannot change by analytic continuation and is thus a phase invariant. Note that unlike the rank, there is no clear physical interpretation of the integers s_+ and s_-. We shall call the set $\{s_+, s_-\}$ the *refined rank*.

It is actually easy to write down explicitly operator relations valid at a given rank or for given $\{s_+, s_-\}$ using the notion of subdiscriminants, see appendix A.

The confinement index. The fact that both R and F, $F'/F = R$, are single valued on the same curve (5.21) implies that the period integrals of the one-form $R\mathrm{d}z$ must be integers. As is well-known, these integers are identified with the integers N_i and k_i that label the vacua $|N_1, k_1; \ldots; N_d, k_d\rangle$ of the theory (these vacua were dicussed in section 2.2.2, example 3).

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Let us now consider the greatest common divisor of the compact periods of $R\mathrm{d}z$ in a given vacuum of rank r for which the integers $N_{i_1}, \ldots, N_{i_r}$ are non-zero,

$$t = N_{i_1} \wedge \cdots \wedge N_{i_r} \wedge (k_{i_1} - k_{i_2}) \wedge \cdots \wedge (k_{i_1} - k_{i_r})\,. \tag{5.26}$$

The periods of $\frac{1}{t}R\mathrm{d}z$ are thus also integers and this implies that not only F but also $F^{1/t}$ will be single-valued on the curve (5.21). Thus there exists an analytic function φ defined on the curve (5.21) such that

$$F(z) = \varphi(z)^t\,. \tag{5.27}$$

Clearly, t cannot change by analytic continuation and is thus a new phase invariant. The integer t can be given a nice physical interpretation [5]: it is the smallest positive integer such that the t^{th} tensor product of the fundamental representation does not confine. For this reason, t is usually called the confinement index. Note that $1 \leq t \leq N$ and that t always divides N.

5.2.3 Semi-classical interpolations

One can, as in 4.2, easily find the possible semiclassical interpolations between the vacua of our model. The quantum effective superpotential is a special case of (4.4)

$$W_{\text{eff}}^{|N_i,k_i\rangle} = \sum_{i=1}^{d} N_i W(w_i) + \sum_{i=1}^{d} N_i \Lambda_i^3 e^{2i\pi k_i/N_i} + \cdots \tag{5.28}$$

with

$$\Lambda_i^{3N_i} = q\frac{W''(w_i)^{N_i}}{\prod_{j\neq i}(w_i - w_j)^{2N_j}}\,. \tag{5.29}$$

These formulas show that:

- the vacua $|N_i, k_i\rangle$ and $|N_i, k_i + 1\rangle$ are smoothly connected at weak coupling by performing the analytic continuation $q \mapsto e^{2i\pi}q$.
- the vacua $|\ldots; N_i, k_i; \ldots; N_j, k_j; \ldots\rangle$ and $|\ldots; N_j, k_j; \ldots; N_i, k_i; \ldots\rangle$ are permuted when w_i and w_j are permuted.

These are the only possible smooth interpolations between vacua at weak coupling.

5.3 The phase diagram

From the above discussion, we can deduce that the ideal of operator relations can be decomposed as

$$\mathscr{I} = \bigcap_{\substack{1\leq s_+,s_-\leq N/2\\ 0\leq s_++s_-\leq N-1}} \ \bigcap_{\substack{1\leq t\leq N\\ t|N}} \mathscr{I}_{\{s_+,s_-\},\,t}\,, \tag{5.30}$$

where $\mathscr{I}_{\{s_+,s_-\},\,t}$ is the ideal of operator relations satisfied in the phases having a given $\{s_+, s_-\}$ and t. It is natural to make the following conjecture.

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Conjecture. There is a unique phase for given refined rank $\{s_+, s_-\}$ and confinement index t. In other words, the ideals $\mathscr{I}_{\{s_+,s_-\},t}$ are prime and (5.30) gives the full phase structure of the model.

It is plausible that a general mathematical proof of this conjecture could be given. Our goal, which is to illustrate in some cases the concepts developed in sections 2 and 3, is more modest and we shall give a proof only when $2 \leq N \leq 7$.

To study the phases at rank r, we always consider a tree level superpotential of degree $d+1 = r+1$. This is the minimal degree that allows the realization of these phases. The phases then also contain the minimal number of vacua (2.19). Considering $d > r$ does not yield any new non-trivial structure; there are more vacua (2.15) but not more phases. The new permutations between vacua that one needs to consider are generated by trivial classical permutations of the roots w_i in (2.11). We shall also always set $g_d = g_r = 1$ for simplicity (this can be achieved by a simple rescaling of the fields).

5.3.1 Some simple cases in general

A few phases can be easily studied for any N.

Phase of rank one. This case can be studied by considering a quadratic tree-level superpotential $W(\phi) = \frac{1}{2}m\phi^2$. There are $\hat{v}_1(N) = N$ vacua $|N,k\rangle$ with unbroken gauge group $\mathrm{U}(N)$ that all have $t = N$ and $\{s_+, s_-\} = \{N/2, N/2-1\}$ is N is even or $\{s_+, s_-\} = \{(N-1)/2, (N-1)/2\}$ is N is odd. It is straightforward to find the explicit solution and to show that $\langle N,k|S|N,k\rangle = mq^{1/N}e^{2i\pi k/N}$. All the vacua are thus trivially related to each other by analytic continuation and thus there is a unique phase at this rank (this also follows from the analysis at weak coupling in 5.2.3). This phase is of course the same as the confining phase of the pure gauge theory (3.13), which can be obtained my sending m to infinity.

Phase of rank N. This is the Coulomb phase with $\hat{v}_N(N) = 1$ vacuum $|1,0;\ldots;1,0\rangle$, $t = 1$ and $\{s_+, s_-\} = \{0,0\}$. The unbroken gauge group is $\mathrm{U}(1)^N$. Note that with one vacuum there can be only one phase. The solution to the constraints (5.19) and (5.20) is simply (since $d = r = N$)

$$W' = g_N P\,, \quad \Delta_S = g_N^2 q\,. \tag{5.31}$$

Phase of rank $N-1$. There are $\hat{v}_{N-1}(N) = 2N-2$ vacua in this case, labeled as $|1,0;\ldots;1,0;2,k;1,0;\ldots;1,0\rangle$, $0 \leq k \leq 1$, with unbroken gauge group $\mathrm{U}(1)^{N-2} \times \mathrm{U}(2)$. All these vacua have $t = 1$ and $\{s_+, s_-\} = \{1,0\}$. There is only one phase because all the vacua can be smoothly related at weak coupling as explained in 5.2.3.

Phases with $\{s_+, s_-\} = \{0, N-r\}$. These phases can exist at ranks $r \geq N/2$. They generalize the phases of rank N and $N-1$ discussed previously. As noticed in [5], the solution to the constraints (5.19), (5.20) and (5.23) has a simple form. One immediately gets $M_- = M_{N-r}$ and

$$\begin{aligned} C_{2r} &= P_+ C_- = (P_- - 4q^{1/2})C_- = (M_{N-r}^2 C_- - 4q^{1/2})C_- \\ &= (M_{N-r}C_-)^2 - 4q^{1/2}C_-\,. \end{aligned} \tag{5.32}$$

Since $d = r$, one also has

$$g_r^2 C_{2r} = W'^2 - 4\Delta_S\,. \tag{5.33}$$

Comparing (5.32) and (5.33), we get

$$(W' - g_r M_{N-r} C_-)(W' + g_r M_{N-r} C_-) = 4(\Delta_S - g_r^2 q^{1/2} C_-)\,. \tag{5.34}$$

Let us assume now that $r \leq N-1$ (the solution for $r = N$ is given by (5.31)). This condition ensures that $\deg C_- = 2r - N \leq r-1$ and thus the degree of the right hand side of (5.34) is less than or equal to $r-1$. Since $\deg(W' + g_r M_{N-r} C_-) = r$, (5.34) implies that

$$W' = g_r M_{N-r} C_-\,, \quad \Delta_S = g_r^2 q^{1/2} C_-\,, \quad P = M_{N-r}^2 C_- - 2q^{1/2}\,. \tag{5.35}$$

The first equation in (5.35) fixes the polynomials M_{N-r} and C_-. There is a $\binom{r}{N-r}$-fold degeneracy corresponding to the choice of the $N-r$ roots of M_{N-r} amongst the r roots of W'. The second equation fixes the glueball operators and adds a twofold degeneracy corresponding to the choice of sign for the square root of q. Overall, the solution thus describes $2\binom{r}{N-r}$ vacua. The third equation fixes the scalar operators and is also very convenient to study the classical limit. The unbroken gauge group is clearly $\mathrm{U}(1)^{2r-N} \times \mathrm{U}(2)^{N-r}$ and, by computing the first semi-classical corrections, it is straightforward to check that the $2\binom{r}{N-r}$ vacua are of the form $|2,k;\ldots;2,k;1,0;\ldots;1,0\rangle$ for $0 \leq k \leq 1$, with $N-r$ slots $2,k$ and $2r-N$ slots $1,0$ that can be permuted in all possible ways. All these vacua can be smoothly connected at weak coupling and thus there is only one phase of this type for any given $r \geq N/2$. Note finally that the confinement index is always $t = 1$, except in the case N even and $r = N/2$ for which $t = 2$.

The full classification of the phases for the gauge groups U(2) and U(3) immediately follows from the above discussion.

- The U(2) theory can have the Coulomb phase of rank two and confinement index one corresponding to the vacuum $|1,0;1,0\rangle$ and the confining phase of rank one and confinement index two corresponding to the vacua $|2,0\rangle$ and $|2,1\rangle$.

- The U(3) theory has the Coulomb phase of rank three and confinement index one (vacuum $|1,0;1,0;1,0\rangle$), the confining phase of rank one and confinement index three (vacua $|3,0\rangle$, $|3,1\rangle$ and $|3,2\rangle$) and the phase of rank two with $\{s_+, s_-\} = \{0,1\}$, $t = 1$ and vacua $|2,0;1,0\rangle$, $|2,1;1,0\rangle$, $|1,0;2,0\rangle$ and $|1,0;2,1\rangle$.

- In the case of U(4), we get immediately the phases at rank four (the Coulomb phase with vacuum $|1,0;1,0;1,0;1,0\rangle$), three (one phase containing the six vacua $|1,0;1,0;2,0\rangle$, $|1,0;1,0;2,1\rangle$ and permutations of the slots) and one (the confining phase with vacua $|4,k\rangle$ for $0 \leq k \leq 3$). At rank two, we have the phase $\{s_+, s_-\} = \{0,2\}$ with $t = 2$ containing the two vacua $|2,0;2,0\rangle$ and $|2,1;2,1\rangle$. There remains eight vacua at rank two, $|2,0;2,1\rangle$, $|2,1;2,0\rangle$, $|3,k;1,0\rangle$ and $|1,0;3,k\rangle$ for $0 \leq k \leq 2$, all having $t = 1$ and $\{s_+, s_-\} = \{1,1\}$. All these vacua were shown to be in the same phase in [8]. This gives the simplest example of a smooth interpolation between different gauge groups, here $\mathrm{U}(2) \times \mathrm{U}(2)$ and $\mathrm{U}(1) \times \mathrm{U}(3)$ [5].

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5.3.2 The case of U(5)

The phases of ranks one, four and five have already been studied in 5.3.1. At rank two, there are 20 vacua all having $\{s_+, s_-\} = \{2, 1\}$, $t = 1$ and unbroken gauge groups $\mathrm{U}(1) \times \mathrm{U}(4)$ (eight vacua) or $\mathrm{U}(2) \times \mathrm{U}(3)$ (twelve vacua). These 20 vacua belong to the same phase as shown in [5, 8].

At rank three, we have six $\mathrm{U}(1) \times \mathrm{U}(2)^2$ vacua in the phase $\{s_+, s_-\} = \{0, 2\}$. The remaining rank three vacua correspond to the other six $\mathrm{U}(1) \times \mathrm{U}(2)^2$ vacua, given by $|2,1;2,0;1,0\rangle$, $|2,0;2,1;1,0\rangle$, $|2,0;1,0;2,1\rangle$, $|2,1;1,0;2,0\rangle$, $|1,0;2,0;2,1\rangle$, $|1,0;2,1;2,1\rangle$, and the nine $\mathrm{U}(1)^2 \times \mathrm{U}(3)$ vacua. These fifteen vacua all have $t = 1$ and $\{s_+, s_-\} = \{1, 1\}$. We have worked out the degree fifteen polynomial equation satisfied by the operator $x = v_0/5$ in these vacua,

$$\begin{aligned} P(x) = x^{15} &+ \left(3qg_1 - qg_2^2\right)x^{12} + 15q^2x^{10} + \left(12q^3g_1 - 4q^3g_2^2\right)x^7 \\ &+ \left(-4g_1^3q^3 - 4g_0g_2^3q^3 - 27g_0^2q^3 + g_1^2g_2^2q^3 + 18g_0g_1g_2q^3\right)x^6 + 48q^4x^5 + \\ &\left(-4g_2^4q^4 - 36g_1^2q^4 + 24g_1g_2^2q^4\right)x^4 + \left(32q^5g_2^2 - 96q^5g_1\right)x^2 - 64q^6 = 0\,. \end{aligned} \quad (5.36)$$

Note that the coefficients of the polynomial are in $\mathbb{C}[g_0, g_1, g_2, q]$ as they should. We have shown using PHC and Singular that P is irreducible over $\mathbb{C}[g_0, g_1, g_2, q]$. This implies that the fifteen vacua under consideration are in the same phase.

5.3.3 The case of U(6)

Again, the phases of rank one, six and five are already known.

Rank two. At rank two, there are 35 vacua that can have either $t = 1$, $t = 2$ or $t = 3$. Thus there must be at least three distinct phases. The three vacua $|3,k;3,k\rangle$, $0 \le k \le 2$ at $t = 3$ are connected semi-classically, and thus must form a unique phase. The eight vacua at $t = 2$ correspond to $\{s_+, s_-\} = \{3, 1\}$ and can all be obtained by semi-classical interpolations starting for example from $|2,0;4,0\rangle$. They are thus also trivially forming a unique phase.

The case of the 24 vacua having $t = 1$ is more interesting. They all have $\{s_+, s_-\} = \{2, 2\}$, so we have studied the polynomial equations satisfied by the chiral operators in this case. In particular, we have found using Singular that when $\{s_+, s_-\} = \{2, 2\}$ the operator $x = v_1/6$ satisfies a degree 27 equation that factorizes into two irreducible pieces of degrees 3 and 24. The degree 3 part is simply $x^3 - q$ and is associated with the $t = 3$

vacua. The degree 24 part is given by

$$\begin{aligned}
P(x) =\,& x^{24} - 8qx^{21} + \left(14qg_1^4 - 61qg_0g_1^2 + 38qg_0^2\right) x^{20} \\
&+ \left(-qg_1^8 + 11qg_0g_1^6 - 41qg_0^2g_1^4 + 56qg_0^3g_1^2 - 16qg_0^4\right) x^{19} + 16q^2x^{18} \\
&+ \left(103q^2g_1^4 - 158q^2g_0g_1^2 - 80q^2g_0^2\right) x^{17} \\
&+ \left(-13q^2g_1^8 + 115q^2g_0g_1^6 - 307q^2g_0^2g_1^4 + 153q^2g_0^3g_1^2 + 367q^2g_0^4\right) x^{16} \\
&+ \big(q^2g_1^{12} - 15q^2g_0g_1^{10} + 87q^2g_0^2g_1^8 - 237q^2g_0^3g_1^6 + 262q^2g_0^4g_1^4 \\
&+ 64q^2g_0^5g_1^2 - 288q^2g_0^6 + 16q^3\big)x^{15} + \big(64q^2g_0^8 - 48q^2g_1^2g_0^7 \\
&+ 12q^2g_1^4g_0^6 - q^2g_1^6g_0^5 - 272q^3g_0^2 + 424q^3g_1^2g_0 + 145q^3g_1^4\big)x^{14} \\
&+ \left(-24q^3g_1^8 + 62q^3g_0g_1^6 + 266q^3g_0^2g_1^4 - 1384q^3g_0^3g_1^2 + 1584q^3g_0^4\right) x^{13} \\
&+ \big(-2q^3g_1^{12} + 26q^3g_0g_1^{10} - 147q^3g_0^2g_1^8 + 519q^3g_0^3g_1^6 - 1390q^3g_0^4g_1^4 \\
&+2518q^3g_0^5g_1^2 - 1740q^3g_0^6 - 56q^4\big)x^{12} + \big(-5q^3g_0^3g_1^{10} + 79q^3g_0^4g_1^8 - 477q^3g_0^5g_1^6 + 1352q^3g_0^6g_1^4 \\
&+282q^4g_1^4 - 1744q^3g_0^7g_1^2 + 792q^4g_0g_1^2 + 768q^3g_0^8 + 240q^4g_0^2\big)x^{11} + \big(-128q^3g_0^{10} + 352q^3g_1^2g_0^9 \\
&- 280q^3g_1^4g_0^8 + 98q^3g_1^6g_0^7 - 16q^3g_1^8g_0^6 + q^3g_1^{10}g_0^5 + 1536q^4g_0^4 - 1412q^4g_1^2g_0^3 \\
&+ 525q^4g_1^4g_0^2 - 129q^4g_1^6g_0 + 47q^4g_1^8\big)x^{10} + \big(q^4g_1^{12} - 16q^4g_0g_1^{10} \\
&+ 114q^4g_0^2g_1^8 - 103q^4g_0^3g_1^6 - 688q^4g_0^4g_1^4 + 2140q^4g_0^5g_1^2 - 2528q^4g_0^6 - 32q^5\big)x^9 \\
&+ \big(10q^4g_0^3g_1^{10} - 113q^4g_0^4g_1^8 + 328q^4g_0^5g_1^6 + 128q^4g_0^6g_1^4 - 49q^5g_1^4 - 1558q^4g_0^7g_1^2 \\
&+ 1076q^5g_0g_1^2 + 1583q^4g_0^8 + 728q^5g_0^2\big)x^8 + \big(-480q^4g_0^{10} + 448q^4g_1^2g_0^9 - 22q^4g_1^4g_0^8 - 75q^4g_1^6g_0^7 \\
&+ 23q^4g_1^8g_0^6 - 2q^4g_1^{10}g_0^5 + 48q^5g_0^4 - 1456q^5g_1^2g_0^3 + 412q^5g_1^4g_0^2 + 232q^5g_1^6g_0 - 10q^5g_1^8\big)x^7 \\
&+ \big(64q^4g_0^{12} - 48q^4g_1^2g_0^{11} + 12q^4g_1^4g_0^{10} - q^4g_1^6g_0^9 - 912q^5g_0^6 + 944q^5g_1^2g_0^5 \\
&- 165q^5g_1^4g_0^4 - 286q^5g_1^6g_0^3 - 27q^5g_1^8g_0^2 + 5q^5g_1^{10}g_0 + 64q^6\big)x^6 \\
&+ \big(-5q^5g_0^3g_1^{10} + 34q^5g_0^4g_1^8 + 68q^5g_0^5g_1^6 + 80q^6g_1^4 - 274q^5g_0^6g_1^4 \\
&- 136q^5g_0^7g_1^2 + 584q^6g_0g_1^2 + 664q^5g_0^8 + 416q^6g_0^2\big)x^5 \\
&+ \big(-186q^5g_0^{10} - 89q^5g_1^2g_0^9 + 128q^5g_1^4g_0^8 - 9q^5g_1^6g_0^7 - 7q^5g_1^8g_0^6 + q^5g_1^{10}g_0^5 \\
&- 296q^6g_0^4 - 1080q^6g_1^2g_0^3 - 350q^6g_1^4g_0^2 - 48q^6g_1^6g_0 + q^6g_1^8\big)x^4 \\
&+ \big(16q^5g_0^{12} + 24q^5g_1^2g_0^{11} - 15q^5g_1^4g_0^{10} + 2q^5g_1^6g_0^9 - 48q^6g_0^6 \\
&+ 632q^6g_1^2g_0^5 + 362q^6g_1^4g_0^4 + 80q^6g_1^6g_0^3 + 64q^7\big)x^3 \\
&+ \left(80q^6g_0^8 - 132q^6g_1^2g_0^7 - 57q^6g_1^4g_0^6 - 53q^6g_1^6g_0^5 + 64q^7g_0^2 + 16q^7g_1^2g_0 - 8q^7g_1^4\right) x^2 \\
&+ \left(-16q^6g_0^{10} + 18q^6g_1^2g_0^9 - 15q^6g_1^4g_0^8 + 13q^6g_1^6g_0^7 - 64q^7g_0^4 - 88q^7g_1^2g_0^3 + 8q^7g_1^4g_0^2\right) x \\
&+ q^6g_0^{12} + 16q^8 + 8q^7g_0^6 - q^6g_0^9g_1^6 + 3q^6g_0^{10}g_1^4 - q^7g_0^4g_1^4 - 3q^6g_0^{11}g_1^2 + 20q^7g_0^5 = 0\,.
\end{aligned} \tag{5.37}$$

This is a rather non-trivial example of a polynomial equation. Its irreducibility, proven using PHC and Singular, implies that the 24 vacua at $t = 1$ form a unique phase. In particular, the eight $t = 1$, $\mathrm{U}(2) \times \mathrm{U}(4)$ vacua, the ten $\mathrm{U}(5) \times \mathrm{U}(1)$ vacua and the six $t = 1$, $\mathrm{U}(3)^2$ vacua can all be smoothly analytically continued into each other.

Rank three. There are $\hat{v}_3(6) = 56$ vacua at rank three. The vacua $|2, 0; 2, 0; 2, 0\rangle$ and $|2, 1; 2, 1; 2, 1\rangle$ form the $\{0, 3\}$ phase with $t = 2$ and unbroken gauge group $\mathrm{U}(2)^3$. There remains 54 vacua that all have $t = 1$ and $\{s_+, s_-\} = \{2, 1\}$, with patterns of gauge symmetry breaking $\mathrm{U}(4) \times \mathrm{U}(1)^2$ (12 vacua), $\mathrm{U}(3) \times \mathrm{U}(2) \times \mathrm{U}(1)$ (36 vacua) and $\mathrm{U}(2)^2$ (6 vacua at $t = 1$). We have been able to show with SINGULAR and PHC that these 54 vacua form a unique phase and that the glueball operator $x = v_0/6$ is primitive. The polynomial equation satisfied by x is of the form

$$P(x) = A_+(x) A_-(x) = 0\,. \tag{5.38}$$

The $A_\pm$ are polynomials of degree 27 that are irreducible over $\mathbb{C}[g_0, g_1, g_2, q^{1/2}]$. The factors A_+ and A_- are permuted into each other when $q^{1/2} \mapsto -q^{1/2}$, making the polynomial P irreducible over $\mathbb{C}[g_0, g_1, g_2, q]$. Explicitly, one has

$$\begin{aligned}
A_+(x) =& x^{27} - \sqrt{q}\left(3g_1 - g_2^2\right) x^{25} + 18qx^{24} - 2q^{3/2}\left(3g_1 - g_2^2\right) x^{22} \\
&- q^{3/2}\left(-27g_0^2 + 2\left(9g_1g_2 - 2g_2^3\right) g_0 + g_1^2\left(g_2^2 - 4g_1\right) - 36\sqrt{q}\right) x^{21} - 6q^2\left(g_2^2 - 3g_1\right)^2 x^{20} \\
&- 86q^{5/2}\left(g_2^2 - 3g_1\right) x^{19} - 3q^{5/2}\left(27g_0^2 + \left(4g_2^3 - 18g_1g_2\right) g_0 + g_1^2\left(4g_1 - g_2^2\right) + 128\sqrt{q}\right) x^{18} \\
&+ 15q^3\left(g_2^2 - 3g_1\right)^2 x^{17} - 164q^{7/2}\left(3g_1 - g_2^2\right) x^{16} \\
&+ q^{7/2}\left(g_2^6 - 9g_1g_2^4 + 21g_1^2g_2^2 - 3g_1^3 + 162g_0^2 - 12g_0\left(9g_1g_2 - 2g_2^3\right) + 684\sqrt{q}\right) x^{15} \\
&- 68q^{9/2}\left(g_2^2 - 3g_1\right) x^{13} \\
&- q^{9/2}\left(2g_2^6 - 18g_1g_2^4 + 47g_1^2g_2^2 - 26g_1^3 + 189g_0^2 - 14g_0\left(9g_1g_2 - 2g_2^3\right) + 576\sqrt{q}\right) x^{12} \\
&- 15q^5\left(g_2^2 - 3g_1\right)^2 x^{11} - 28q^{11/2}\left(g_2^2 - 3g_1\right) x^{10} \\
&+ q^{11/2}\left(g_2^6 - 9g_1g_2^4 + 21g_1^2g_2^2 - 3g_1^3 + 162g_0^2 - 12g_0\left(9g_1g_2 - 2g_2^3\right) + 336\sqrt{q}\right) x^9 \\
&+ 6q^6\left(g_2^2 - 3g_1\right)^2 x^8 - 28q^{13/2}\left(3g_1 - g_2^2\right) x^7 \\
&- 3q^{13/2}\left(27g_0^2 + \left(4g_2^3 - 18g_1g_2\right) g_0 + g_1^2\left(4g_1 - g_2^2\right) + 48\sqrt{q}\right) x^6 \\
&- 16q^{15/2}\left(g_2^2 - 3g_1\right) x^4 - q^{15/2}\left(-27g_0^2 + 2\left(9g_1g_2 - 2g_2^3\right) g_0 + g_1^2\left(g_2^2 - 4g_1\right) - 36\sqrt{q}\right) x^3 \\
&- 4q^{17/2}\left(3g_1 - g_2^2\right) x - 8q^9\,.
\end{aligned} \tag{5.39}$$

Rank four. Of the 36 vacua at rank four, 12 belong to the $\{0, 2\}$ phase with unbroken gauge group $\mathrm{U}(1)^2 \times \mathrm{U}(2)^2$. These vacua can all be obtained by semi-classical interpolations starting for example from $|1, 0; 1, 0; 2, 0; 2, 0\rangle$. The non-trivial case concerns the 24 vacua having $\{s_+, s_-\} = \{1, 1\}$. There are twelve $\mathrm{U}(3) \times \mathrm{U}(1)^3$ vacua and twelve $\mathrm{U}(1)^2 \times \mathrm{U}(2)^2$ vacua of this sort. We have shown with PHC and SINGULAR that they all belong to the same phase, the glueball operator $x = v_0/6$ being primitive with degree 24 irreducible equation given by

$$
\begin{aligned}
P(x) =& x^{24} + \left(qg_3^3 - 4qg_2g_3 + 8qg_1\right) x^{21} \\
&+ \left(4q^2g_2^3 - q^2g_3^2g_2^2 - 16q^2g_0g_2 - 14q^2g_1g_3g_2 + 3q^2g_1g_3^3 + 18q^2g_1^2 + 6q^2g_0g_3^2\right) x^{18} \\
&+ \left(15q^3g_3^4 - 80q^3g_2g_3^2 + 56q^3g_1g_3 + 88q^3g_2^2 - 224q^3g_0\right) x^{16} + \left(640q^4g_2 - 240q^4g_3^2\right) x^{14} \\
&+ \big(12g_1g_3^4q^4 - 4g_2^2g_3^3q^4 - 48g_0g_3^3q^4 - 32g_1g_2^2q^4 \\
&- 48g_1g_2g_3^2q^4 - 384g_0g_1q^4 + 16g_2^3g_3q^4 + 96g_1^2g_3q^4 + 192g_0g_2g_3q^4\big)x^{13} \\
&+ \big(2176q^5 - 27g_1^4q^4 + 16g_0g_2^4q^4 - 27g_0^2g_3^4q^4 + 256g_0^3q^4 - 4g_1^2g_2^3q^4 - 4g_1^3g_3^3q^4 \\
&+ 18g_0g_1g_2g_3^3q^4 - 128g_0^2g_2^2q^4 - 4g_0g_2^3g_3^2q^4 - 6g_0g_1^2g_3^2q^4 + g_1^2g_2^2g_3^2q^4 \\
&+ 144g_0^2g_2g_3^2q^4 + 144g_0g_1^2g_2q^4 - 80g_0g_1g_2^2g_3q^4 - 192g_0^2g_1g_3q^4 + 18g_1^3g_2g_3q^4\big)x^{12} \\
&+ \left(48g_3^5q^5 - 320g_2g_3^3q^5 + 384g_1g_3^2q^5 - 1024g_1g_2q^5 + 512g_2^2g_3q^5\right) x^{11} \\
&+ \big(16g_2^5q^5 - 36g_1^2g_3^4q^5 + 128g_0g_2^3q^5 + 24g_1g_2^2g_3^3q^5 + 72g_0g_1g_3^3q^5 \\
&+ 864g_0g_1^2q^5 + 72g_1^2g_2^2q^5 - 4g_2^4g_3^2q^5 + 288g_0^2g_3^2q^5 - 24g_0g_2^2g_3^2q^5 \\
&+ 168g_1^2g_2g_3^2q^5 - 768g_0^2g_2q^5 - 216g_1^3g_3q^5 - 104g_1g_2^3g_3q^5 - 480g_0g_1g_2g_3q^5\big)x^{10} \\
&+ \left(-448g_3^3q^6 - 3584g_1q^6 + 1792g_2g_3q^6\right) x^9 \\
&+ \big(-96g_1g_3^5q^6 + 272g_2^4q^6 + 32g_2^2g_3^4q^6 + 96g_0g_3^4q^6 + 608g_1g_2g_3^3q^6 - 1792g_0^2q^6 + 384g_0g_2^2q^6 \\
&- 192g_2^3g_3^2q^6 - 400g_1^2g_3^2q^6 - 512g_0g_2g_3^2q^6 + 768g_1^2g_2q^6 - 864g_1g_2^2g_3q^6 + 896g_0g_1g_3q^6\big)x^8 \\
&+ \big(-64g_3^6q^7 + 512g_2g_3^4q^7 + 1792g_2^3q^7 + 512g_1g_3^3q^7 \\
&+ 1664g_1^2q^7 - 1536g_2^2g_3^2q^7 - 384g_0g_3^2q^7 + 1024g_0g_2q^7 - 1920g_1g_2g_3q^7\big)x^6 \\
&+ \left(768g_3^4q^8 + 5632g_2^2q^8 - 4096g_2g_3^2q^8 + 2048g_0q^8 - 512g_1g_3q^8\right) x^4 \\
&+ \left(8192q^9g_2 - 3072q^9g_3^2\right) x^2 + 4096q^{10} = 0\,.
\end{aligned}
\tag{5.40}
$$

5.3.4 The case of U(7)

This is the most complex case that we are going to study. Note that because $N = 7$ is prime, all the phases with $r \geq 2$ have $t = 1$. Again, the phases of rank one, six and seven, as well as some phases at ranks four and five, have been studied in 5.3.1.

Rank two. At rank two, we have twelve $\mathrm{U}(1) \times \mathrm{U}(6)$ vacua, twenty $\mathrm{U}(2) \times \mathrm{U}(5)$ vacua and twenty-four $\mathrm{U}(3) \times \mathrm{U}(4)$ vacua, for a total of 56 vacua. All these vacua have the same phase invariants: $r = 2$, $t = 1$, $\{s_+, s_-\} = \{3, 2\}$. One thus could expect to have a unique phase containing all these vacua. We have found the degree 56 polynomial equation satisfied by the glueball operator $x = v_0/7$. It has the form (5.38), where now the factors $A_\pm$ are polynomials of degree 28 over $\mathbb{C}[g_0, g_1, q^{1/2}]$ that are permuted when $q^{1/2} \mapsto -q^{1/2}$. Explicitly,

$$
\begin{aligned}
A_+(x) =& x^{28} + 2\sqrt{q}\left(g_1^2 - 4g_0\right) x^{25} + q\left(g_1^2 - 4g_0\right)^2 x^{22} + 46q^{3/2}x^{21} + 4q^2\left(4g_0 - g_1^2\right) x^{18} \\
&- 2q^{5/2}\left(g_1^2 - 4g_0\right)^2 x^{15} - 21q^3x^{14} + q^3\left(4g_0 - g_1^2\right)^3 x^{12} + 36q^{7/2}\left(4g_0 - g_1^2\right) x^{11} \\
&+ 3q^4\left(g_1^2 - 4g_0\right)^2 x^8 + 102q^{9/2}x^7 + 3q^5\left(4g_0 - g_1^2\right) x^4 + q^6\,.
\end{aligned}
\tag{5.41}
$$

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We have shown using PHC and SINGULAR that A_+ is irreducible over $\mathbb{C}[g_0, g_1, q^{1/2}]$, which implies immediately that A_+A_- is irreducible over $\mathbb{C}[g_0, g_1, q]$: the 56 vacua are indeed in the same phase.

Rank three. Since there are 126 rank three vacua, all the chiral operators satisfy a polynomial equation of degree 126 with coefficients in $\mathbb{C}[g_0, g_1, g_2, q]$. We have found this equation for various chiral operators. In particular, we have shown with SINGULAR that the equation for the glueball operator $x = v_0/7$ factorizes into two irreducible pieces of degree 42 and 84 associated with two phases $|42)$ and $|84)$,

$$P(x) = P_{42}(x)P_{84}(x) = 0\,. \tag{5.42}$$

Moreover, P_{42} factorizes over $\mathbb{C}[g_0, g_1, g_2, q^{1/2}]$ into two degree 21 factors A_+ and A_- that are exchanged under $q^{1/2} \mapsto -q^{1/2}$. This shows that P_{42} corresponds to the $\{s_+, s_-\} = \{3, 1\}$ vacua, which therefore must all be in $|42)$. The other 84 vacua thus all have $\{s_+, s_-\} = \{2, 2\}$ and must all be in $|84)$.

It is easy to identify the possible unbroken gauge groups in each phase, for example by looking at the classical limit of the polynomial equations for the operators u_k. It is more difficult to compute the integers k_i for each vacua of the form $|N_1, k_1; N_2, k_2; N_3, k_3\rangle$ in a given phase. To do so, we have computed numerically the gluino condensates s_i in the unbroken $\mathrm{U}(N_i)$ factors of the gauge group by computing the relevant contour integrals of the generating function $S(z)$ given in (5.11). This calculation, that must be repeated in each individual vacua, can be easily implemented on Mathematica. The integers k_i can then be extracted from the small q behaviour $s_i \simeq \Lambda_i^3 e^{2i\pi k_i/N_i}$, where Λ_i is given by (5.29). One can also extract the k_i from some contour integrals of the generating function $R(z)$ (see (5.10)), and we have double-checked the results in this way.

It turns out that the phase $|42)$ contains the twenty-four $\mathrm{U}(1) \times \mathrm{U}(2) \times \mathrm{U}(4)$ vacua that can be obtained from $|1, 0; 2, 0; 4, 0\rangle$ by semi-classical interpolations. It also contains the eighteen $\mathrm{U}(2)^2 \times \mathrm{U}(3)$ vacua that can be obtained from $|2, 0; 2, 0; 3, 0\rangle$ by semi-classical interpolations. For completeness, we also give the formula for the degree 21 polynomial A_+ in this case,

$$\begin{aligned} A_+(x) =& x^{21} + 2q\left(3g_1 - g_2^2\right)x^{17} - 3q^2x^{14} + q^2\left(g_2^2 - 3g_1\right)^2 x^{13} \\ &+ q^3\left(g_2^2 - 3g_1\right)x^{10} + q^3\left(27g_0^2 + \left(4g_2^3 - 18g_1g_2\right)g_0 + g_1^2\left(4g_1 - g_2^2\right)\right)x^9 \\ &- 3q^{7/2}\left(2g_2^3 - 9g_1g_2 + 27g_0\right)x^8 + 57q^4x^7 + q^5\left(g_2^2 - 3g_1\right)x^3 - q^6\,. \end{aligned} \tag{5.43}$$

The phase $|84)$ contains all the rank three vacua that are not in $|42)$, which includes twenty-four $\mathrm{U}(1) \times \mathrm{U}(2) \times \mathrm{U}(4)$ vacua, eighteen $\mathrm{U}(2)^2 \times \mathrm{U}(3)$ vacua, fifteen $\mathrm{U}(1)^2 \times \mathrm{U}(5)$ vacua and twenty-seven $\mathrm{U}(1) \times \mathrm{U}(3)^2$ vacua. The polynomial P_{84} is extremely complicated. It turns out that if we set $g_1 = g_2 = 0$, the polynomial remains irreducible (this of course implies that the polynomial is irreducible in the general case). It is thus enough to present

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P_{82} in this special case,

$$
\begin{aligned}
P(x) =&x^{84} - 81qg_0^2x^{79} + 3699q^2x^{77} + 2187q^2g_0^4x^{74} + 254367q^3g_0^2x^{72} + 3413310q^4x^{70}\\
&- 19683q^3g_0^6x^{69} - 667035q^4g_0^4x^{67} - 7708608q^5g_0^2x^{65} - 13620477q^6x^{63} - 708588q^5g_0^6x^{62}\\
&- 5226930q^6g_0^4x^{60} + 43654221q^7g_0^2x^{58} - 1062882q^6g_0^8x^{57} - 70179075q^8x^{56}\\
&- 35783694q^7g_0^6x^{55} - 496822977q^8g_0^4x^{53} - 2358810882q^9g_0^2x^{51} - 20726199q^8g_0^8x^{50}\\
&- 1698777354q^{10}x^{49} - 193523256q^9g_0^6x^{48} + 940766481q^{10}g_0^4x^{46} - 14348907q^9g_0^{10}x^{45}\\
&- 505521243q^{11}g_0^2x^{44} - 117979902q^{10}g_0^8x^{43} - 4394981908q^{12}x^{42} - 117920853q^{11}g_0^6x^{41}\\
&- 1101683754q^{12}g_0^4x^{39} + 19938963645q^{13}g_0^2x^{37} - 129671604q^{12}g_0^8x^{36}\\
&- 20347899486q^{14}x^{35} - 814226661q^{13}g_0^6x^{34} + 4435334415q^{14}g_0^4x^{32}\\
&- 8334566253q^{15}g_0^2x^{30} + 2904592227q^{16}x^{28} + 277451568q^{15}g_0^6x^{27} - 866557197q^{16}g_0^4x^{25}\\
&+ 684971721q^{17}g_0^2x^{23} + 154884143q^{18}x^{21} - 20016153q^{18}g_0^4x^{18} + 62696268q^{19}g_0^2x^{16}\\
&- 24397098q^{20}x^{14} + 367389q^{21}g_0^2x^9 - 9885q^{22}x^7 - q^{24}\,. \qquad (5.44)
\end{aligned}
$$

Rank four. At rank four, we have a simple $\{3,0\}$ phase which contains the eight vacua obtained from $|2,0;2,0;2,0;1,0\rangle$ by semi-classical interpolations. The other 112 vacua all have $\{s_+,s_-\}=\{2,1\}$. The glueball operator $x=v_0/7$ satisfies a degree 112 irreducible equation of the form (5.38), where now A_+ is of degree 56. It turns out that P remains irreducible is we set $g_1=g_2=g_3=0$, so we can restrict ourselves to this case for which

$$
\begin{aligned}
A_+(x)=&x^{56} + 304q^{3/2}g_0x^{51} - 5532q^{5/2}x^{49} + 256q^2g_0^3x^{48} - 4000q^3g_0^2x^{46} + 27312q^4g_0x^{44}\\
&+ 128262q^5x^{42} + 768q^{9/2}g_0^3x^{41} - 64064q^{11/2}g_0^2x^{39} - 301616q^{13/2}g_0x^{37} + 8448q^6g_0^4x^{36}\\
&- 674364q^{15/2}x^{35} + 92928q^7g_0^3x^{34} + 100704q^8g_0^2x^{32} - 667440q^9g_0x^{30} + 24576q^{17/2}g_0^4x^{29}\\
&- 13439q^{10}x^{28} - 111616q^{19/2}g_0^3x^{27} - 1355520q^{21/2}g_0^2x^{25} + 73728q^{10}g_0^5x^{24}\\
&- 582656q^{23/2}g_0x^{23} + 307200q^{11}g_0^4x^{22} + 247776q^{25/2}x^{21} - 747008q^{12}g_0^3x^{20}\\
&- 1274368q^{13}g_0^2x^{18} + 196608q^{25/2}g_0^5x^{17} - 531328q^{14}g_0x^{16} - 65536q^{27/2}g_0^4x^{15}\\
&- 179840q^{15}x^{14} - 679936q^{29/2}g_0^3x^{13} + 65536q^{14}g_0^6x^{12} - 679936q^{31/2}g_0^2x^{11}\\
&+ 131072q^{15}g_0^5x^{10} - 411648q^{33/2}g_0x^9 + 98304q^{16}g_0^4x^8 - 149504q^{35/2}x^7\\
&+ 65536q^{17}g_0^3x^6 + 36864q^{18}g_0^2x^4 + 8192q^{19}g_0x^2 + 4096q^{20}\,. \qquad (5.45)
\end{aligned}
$$

Thus we can interpolate smoothly between the twenty-four $\mathrm{U}(1)\times\mathrm{U}(2)^3$ vacua, sixteen $\mathrm{U}(1)^3\times\mathrm{U}(4)$ vacua and seventy-two $\mathrm{U}(1)^2\times\mathrm{U}(2)\times\mathrm{U}(3)$ vacua of the phase.

Rank five. The twenty vacua that can be obtained by semi-classical interpolations from $|2,0;2,0;1,0;1,0;1,0\rangle$ form the phase $\{s_+,s_-\}=\{0,2\}$. The remaining thirty-five vacua (twenty $\mathrm{U}(1)^3\times\mathrm{U}(2)^2$ and fifteen $\mathrm{U}(1)^4\times\mathrm{U}(3)$) all have $\{s_+,s_-\}=\{1,1\}$ and form a unique phase. Indeed, $x=v_0/7$ satisfies a degree 35 polynomial equation. For $g_4=0$ (this can always be achieved by a simple shift in the tree-level superpotential), this equation reads

$$
\begin{aligned}
P(x) =& x^{35} - q\big(4g_3^2 - 15g_1\big)x^{32} - 65q^2 g_3 x^{30} \\
& - q^2\big(-4g_3^4 + 36g_1 g_3^2 - 27g_2^2 g_3 - 80g_1^2 + 50g_0 g_2\big)x^{29} - 705q^3 x^{28} \\
& + 100q^3\big(g_3^3 - 4g_1 g_3 + 4g_2^2\big)x^{27} - q^3\big(27g_2^4 + 4g_3^3 g_2^2 + 40g_0 g_3^2 g_2 - 160g_1^3 + 88g_1^2 g_3^2 \\
& - 125g_0^2 g_3 + 3g_1\big(-4g_3^4 - 39g_2^2 g_3 + 100g_0 g_2\big)\big)x^{26} - 4q^4\big(1140g_1 - 529g_3^2\big)x^{25} \\
& - q^4\big(-60g_3^5 - 353g_2^2 g_3^2 + 160g_1^2 g_3 + 3350g_0 g_2 g_3 - 3750g_0^2 - 2g_1\big(705g_2^2 - 92g_3^3\big)\big)x^{24} \\
& + 21760q^5 g_3 x^{23} - 4q^5\big(760g_1^2 + 1158g_3^2 g_1 + 6650g_0 g_2 - 47\big(9g_3^4 + 28g_2^2 g_3\big)\big)x^{22} \\
& + 4q^5\big(12g_1 g_3^5 - 4g_2^2 g_3^4 - 128g_1^2 g_3^3 + 117g_1 g_2^2 g_3^2 - 27g_2^4 g_3 + 320g_1^3 g_3 - 180g_1^2 g_2^2 \\
& + 30640q + 625g_0^2\big(4g_1 - g_3^2\big) + 10g_0 g_2\big(8g_3^3 - 40g_1 g_3 + 45g_2^2\big)\big)x^{21} \\
& - q^5\big(256g_1^5 - 128g_3^2 g_1^4 + 16\big(g_3^4 + 9g_2^2 g_3\big)g_1^3 - \big(27g_2^4 + 4g_3^3 g_2^2\big)g_1^2 + 62400q g_3 g_1 + 3125g_0^4 \\
& - 3750g_0^3 g_2 g_3 + 80q\big(5g_2^2 - 306g_3^3\big) + g_0^2\big(108g_3^5 + 825g_2^2 g_3^2 + 2000g_1^2 g_3 + 450g_1(5g_2^2 - 2g_3^3)\big) \\
& + 2g_0 g_2\big(54g_2^4 + 8g_3^3 g_2^2 - 800g_1^3 + 280g_1^2 g_3^2 - 9g_1\big(4g_3^4 + 35g_2^2 g_3\big)\big)\big)x^{20} \\
& + 16q^6\big(880g_1^3 - 184g_3^2 g_1^2 + \big(-73g_3^4 + 116g_2^2 g_3 + 600g_0 g_2\big)g_1 + 3\big(4g_3^6 + 32g_2^2 g_3^3 + 10g_0 g_2 g_3^2 \\
& - 500g_0^2 g_3 + 33g_2^4\big)\big)x^{19} - 4q^6\big(27g_2^6 + 4g_3^3 g_2^4 - 198g_1 g_3 g_2^4 - 24g_1 g_3^4 g_2^2 - 40g_1^3 g_2^2 + 434g_1^2 g_3^2 g_2^2 \\
& - 6250g_0^3 g_2 + 10g_0\big(-33g_1 g_3^3 + 6g_2^2 g_3^2 + 20g_1^2 g_3 - 45g_1 g_2^2\big)g_2 + 36g_1^2 g_3^5 - 224g_1^3 g_3^3 \\
& + 320g_1^4 g_3 + 32q\big(2095g_1 - 1292g_3^2\big) + 25g_0^2\big(27g_3^4 - 150g_1 g_3^2 + 120g_2^2 g_3 + 200g_1^2\big)\big)x^{18} \\
& - 64q^7\big(-67g_3^5 + 558g_1 g_3^3 - 436g_2^2 g_3^2 - 680g_1^2 g_3 - 300g_0 g_2 g_3 + 1875g_0^2 - 95g_1 g_2^2\big)x^{17} \\
& + 16q^7\big(-400g_1^4 + 360g_3^2 g_1^3 - 5\big(-3g_3^4 + 192g_2^2 g_3 + 200g_0 g_2\big)g_1^2 + 2\big(-12g_3^6 \\
& - 70g_2^2 g_3^3 + 825g_0 g_2 g_3^2 + 2500g_0^2 g_3 + 135g_2^4\big)g_1 - 40g_0 g_2 g_3\big(g_3^3 + 15g_2^2\big) \\
& - 125g_0^2\big(14g_3^3 + 15g_2^2\big) + g_3\big(45g_3 g_2^4 + 8g_3^4 g_2^2 + 28960q\big)\big)x^{16} \\
& + 256q^8\big(129g_3^4 - 988g_1 g_3^2 + 736g_2^2 g_3 + 640g_1^2 + 350g_0 g_2\big)x^{15} \\
& + 64q^8\big(-4g_3^7 - 106g_1 g_3^5 + 7g_2^2 g_3^4 + 308g_1^2 g_3^3 - 612g_1 g_2^2 g_3^2 + 102g_2^4 g_3 + 80g_1^3 g_3 - 570g_1^2 g_2^2 \\
& + 9360q + 1250g_0^2\big(3g_1 - 2g_3^2\big) - 10g_0 g_2\big(8g_3^3 - 215g_1 g_3 + 70g_2^2\big)\big)x^{14} \\
& + 10240q^9\big(11g_3^3 - 75g_1 g_3 + 40g_2^2\big)x^{13} - 256q^9\big(-53g_2^4 + 52g_3^3 g_2^2 + 240g_0 g_3^2 g_2 \\
& + 40g_1^3 - 522g_1^2 g_3^2 + g_1\big(194g_3^4 + 708g_2^2 g_3 - 950g_0 g_2\big) + 25\big(g_3^6 + 70g_0^2 g_3\big)\big)x^{12} \\
& - 4096q^{10}\big(235g_1 - 46g_3^2\big)x^{11} \\
& - 1024q^{10}\big(66g_3^5 + 118g_2^2 g_3^2 - 360g_1^2 g_3 + 400g_0 g_2 g_3 + 625g_0^2 + 4g_1\big(49g_3^3 + 65g_2^2\big)\big)x^{10} \\
& + 245760q^{11} g_3 x^9 - 4096q^{11}\big(95g_3^4 + 124g_1 g_3^2 + 92g_2^2 g_3 - 95g_1^2 + 200g_0 g_2\big)x^8 + 327680q^{12} x^7 \\
& - 81920q^{12}\big(16g_3^3 + 10g_1 g_3 + 5g_2^2\big)x^6 \\
& - 65536q^{13}\big(39g_3^2 + 10g_1\big)x^4 - 2621440q^{14} g_3 x^2 - 1048576q^{15} = 0\,.
\end{aligned}
\tag{5.46}
$$

The above polynomial can be shown to be irreducible over $\mathbb{C}[g_0, g_1, g_2, g_3, q]$ using both PHC and Singular. Let us spell out, for the last time, the two basic consequences of the irreducibility. First, the 35 vacua that correspond to the 35 roots of the polynomial can all be smoothly connected to each other by analytic continuations in the parameters. Second, the operator x, or v_0, is a primitive operator. Thus *any* chiral operator in *any* of the 35 vacua of the phase is given by a simple polynomial in x.

N	1	2	3	4	5	6	7
v	1	5	22	101	476	2282	11075
φ	1	2	3	5	6	10	10

Table 3: Number of vacua and of phases for various values of N.

5.3.5 Summary

In table 3 we give, for each value of N, the total number v of vacua and the total number φ of distinct phases in the model $d = N$, which is the simplest model that realizes all the possible phases.

6. Conclusion

In the present paper, we have used the language of algebraic geometry, at an elementary level, to formulate and analyse the exact solutions to $\mathcal{N} = 1$ supersymmetric gauge theories. We have demonstrated that this approach is completely general and has many practical advantages. It eliminates confusing points appearing in other approaches, allows for an elegant global description of the quantum phases and can be efficiently implemented on the computer. It also provides a precise formulation of Seiberg dualities. We believe that this is the most appropriate language in which to discuss the quantum supersymmetric theories.

Of course there are many possible applications of the formalism and many open problems could be fruitfully studied along the lines of our work. An outstanding example is the $\mathcal{N} = 1^*$ theory, which is a deformation of $\mathcal{N} = 4$ in which supersymmetry is broken down to $\mathcal{N} = 1$ by turning on a tree-level superpotential for the three adjoints X, Y and Z of the form $\frac{1}{2}\,\mathrm{Tr}(m_Y Y^2 + m_Z Z^2 + V(X))$, where V is an arbitrary polynomial. Almost nothing is known about the phase structure of this model beyond the case of the massive phases [28], which are the analogues of the rank one phases studied in 5.3.1. A particularly interesting feature of the $\mathcal{N} = 1^*$ model is that it inherits the S-duality of the $\mathcal{N} = 4$ theory and thus the S-duality group has a non-trivial action on the vacua of the theory.

Another important problem that we have only skimmed over in 3.6 is the study of the possible phase transitions. Phase transitions can be associated with non-trivial superconformal fixed points and an interesting physics. For example, standard cases involve the condensation of monopoles, and many more exotic phenomena can be expected. The methods of the present paper are very well suited to make a systematic study of these transitions, for example in the models that we have discussed in sections 4 and 5.

Another very natural arena to apply our methods is the landscape of supersymmetric vacua in string or M theory. Can we find in this context simple models where a full analysis can be performed? What are the irreducible components of the space of vacua? Can we obtain a full description of the possible phase transitions? What is the rôle played by gravity in shaping the structure of the phase diagram? What are the consequences of the existence of distinct phases (as opposed to distinct vacua) when one tries to use statistical methods to study the landscape?

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An important lesson that we have learned is that the notion of phase is a much more fundamental concept than the notion of vacuum in a fully quantum treatment of the supersymmetric theories. The phases are the basic, irreducible, building blocks of the quantum theory. This has interesting consequences for the landscape of possible universes. For example, the existence of a given vacuum implies, by quantum consistency, the existence of all the other vacua in the same phase. In our framework, this simply follows from the fact that the semiclassical expansion of any given root of an irreducible polynomial characterizes completely the irreducible polynomial and thus all the other roots.

Another interesting remark is that it is clearly much more convenient and natural to work with the irreducible polynomials themselves than with the series expansions. This feature is in tension with the standard approach to quantum theory based on the quantization of classical systems and suggests that a better formulation of quantum theory might exist.

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A. On subdiscriminants

It is possible to write down explicit operator relations valid for given values of the rank r or of the integers s_+ and s_- that correspond to the factorization conditions (5.19) or (5.23). The general problem is as follows: given a certain polynomial

$$H(z) = \sum_{k=0}^{n} c_k z^{n-k} = c_0 \prod_{i=1}^{n} (z - h_i)\,, \tag{A.1}$$

what are the conditions on the coefficients c_k for Q to have p double roots? The answer to this question, in the case $p = 1$, is well-known. One introduces the discriminant of H,

$$\Delta_H^{(0)} = c_0^{n-1} \prod_{i<j} (h_i - h_j)^2\,. \tag{A.2}$$

Clearly, $\Delta_H^{(0)} = 0$ if and only if H has a double root. Moreover, $\Delta_H^{(0)}$ is a symmetric polynomial in the roots h_i and can thus be written as a polynomial in the coefficients c_k. The algebraic equation

$$\Delta_H^{(0)}(c_0, \ldots, c_n) = 0 \tag{A.3}$$

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gives the necessary and sufficient condition for H to have a double root.

For example, the ideal corresponding to the rank $N-1$ vacua is generated by the polynomial $\Delta^{(0)}_{P^2-4q}$. In the notation of (5.30), this ideal corresponds to $\mathscr{I}_{\{N/2,N/2-1\},1}$ if N is even or to $\mathscr{I}_{\{(N-1)/2,(N-1)/2\},1}$ if N is odd. As explained in 5.3, these ideals are prime, and thus the polynomials $\Delta^{(0)}_{P^2-4q}$ are irreducible.

Assume now that H has one double root. Can we find an additional condition on the coefficients c_k that would ensure that H actually has two double roots (or one triple root)? This condition is not difficult to guess. Consider

$$\Delta^{(1)}_H = c_0^{n-2} \sum_{k=1}^{n} \prod_{\substack{i<j \\ i,j\neq k}} (h_i - h_j)^2 \,. \tag{A.4}$$

If, for example, $h_1 = h_2$, then $\Delta^{(1)}_H = c_0^{n-2}\prod_{2\leq i<j}(h_i-h_j)^2$. Imposing $\Delta^{(1)}_H = 0$ thus clearly does the job. Note also that $\Delta^{(1)}$ is completely symmetric in the roots and can be expressed as a polynomial in the coefficients as required. More generally, one has the following standard definitions and theorems.

Definition 11. The k^{th} *subdiscriminant*, $0 \leq k \leq n-2$, of the polynomial H in (A.1) is defined by

$$\Delta^{(k)}_H = c_0^{n-k-1} \sum_{\substack{I\subset\{1,\ldots,n\} \\ |I|=n-k}} \prod_{\substack{i<j \\ (i,j)\in I^2}} (h_i - h_j)^2 \,, \tag{A.5}$$

where the sum in the right hand side of (A.5) has $\binom{n}{k}$ terms, running over all subsets $I \subset \{1,\ldots,n\}$ of cardinality $n-k$.

Proposition 21. *The k^{th} subdiscriminant of H is a polynomial in the coefficients of H. Explicitly, if we denote by $N_j = \sum_{i=1}^n h_i^j$ the j^{th} Newton's sum and by $\mathcal{H}^{(k)} = (N_{i+j-2})_{1\leq i,j\leq n-k}$ the k^{th} Hermite's matrix, then*

$$\Delta^{(k)}_H(c_0,\ldots,c_n) = c_0^{n-k-1} \det \mathcal{H}^{(k)} \,. \tag{A.6}$$

Theorem 22. *The polynomial H in* (A.1) *has p double roots (where a q^{th} root is counted as $q-1$ double roots) if and only if the algebraic equations*

$$\Delta^{(0)}_H = \cdots = \Delta^{(p-1)}_H = 0 \tag{A.7}$$

on its coefficients are satisfied.

Proposition 21 can be derived by noting that, if

$$\mathcal{V}^{(k)} = (h_j^{i-1})_{\substack{1\leq i\leq n-k \\ 1\leq j\leq n}} \tag{A.8}$$

is a truncated Van der Monde matrix, then $\mathcal{H}^{(k)} = \mathcal{V}^{(k)T}\mathcal{V}^{(k)}$. One then uses the Cauchy-Binet formula for the determinant of the product of two matrices and the standard result for the Van der Monde determinants to obtain (A.6). Theorem 22 follows directly from the definition (A.4).

For a given rank r, the operator relations

$$\Delta^{(0)}_{P^2-4q} = \cdots = \Delta^{(N-r-1)}_{P^2-4q} = 0 \tag{A.9}$$

are thus satisfied. For given $\{s_+, s_-\}$, one has the relations

$$\Delta^{(0)}_{P_+} = \cdots = \Delta^{(s_+-1)}_{P_+} = 0 = \Delta^{(0)}_{P_-} = \cdots = \Delta^{(s_--1)}_{P_-}\,. \tag{A.10}$$

These are not operator relations in the strict sense because $P_\pm = P \mp 2q^{1/2}$ and thus $q^{1/2}$ enters in the coefficients, but any combination of the relations (A.10) that is invariant under $q^{1/2} \mapsto -q^{-1/2}$ (or equivalently under the interchange of P_+ and P_-) will be a proper operator relation.

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